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2.) Surface temperature of the Sun.

The value of the total radiant energy flux density at the Earth from the Sun normal to the incident rays is called the solar constant of the earth. The observed value integrated over all emission wavelengths and referred to the mean Earth-Sun distance is:

Solar constant = 0.136 J s^{-1} cm^{-2}

(a) Show that the total rate of energy generation of the Sun is 4×10^{26} J s^{-1}.

$$P_{sun} = S\left(4\pi R^2_{earth-sun}\right) = \left(0.136 Js^{-1}cm^{-2}\right)\left(4\pi\right)\left(1.5\cdot10^{13}cm\right)^2 = 4\cdot10^{26}Js^{-1}$$

(b) From this result and the Stefan-Boltzmann constant show that the effective temperature of the surface of the Sun treated as a blackbody is T~6000K. Take the distance of the Earth from the Sun as 1.5×10^{13} cm and the radius of the Sun as 7×10^{10} cm.

$$\frac{P_{sun}}{Area} = \sigma_B T^4$$

$$T = \left(\frac{P_{sun}}{\sigma_B\left(4\pi\right)R^2_{sun}}\right)^{1/4} = \left(\frac{4\cdot10^{26}Js^{-1}}{5.67\cdot10^{-12}Js^{-1}cm^{-2}K^{-4}\left(4\pi\right)\left(7\cdot10^{10}cm\right)^2}\right)^{1/4}$$

$$T \sim 6000K$$

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4.) Average temperature of the interior of the Sun.

(a) Estimate by a dimensional argument or otherwise, the order of magnitude of the gravitational self-energy of the Sun, with M=2×10^{33} g and R=7×10^{10} cm. The gravitational constant G is 6.67×10^{-8} dyne cm^2 g^{-2}. The self-energy will be negative referred to atoms at rest at infinite separation.

$$U_{Gravity} = \tfrac{1}{2}\sum_i^{natoms}\sum_j^{natoms}\frac{Gm_i m_j}{\left|\vec{r}_i - \vec{r}_j\right|}$$

By analogy to systems of charges and the energy of a sphere of uniform charge, the gravitational potential (estimating using a uniform mass density for the Sun):

$$V_{Gravity} = -G\frac{\rho\frac{4}{3}\pi R^3_{Sun}}{r} = -G\frac{M_{Sun}}{r} \qquad r > R_{Sun}$$

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$$V_{Gravity}=-G\rho\int_0^{2\pi}\int_0^{\pi}\int_0^{R_{Sun}}\frac{r'^2\sin\theta' dr' d\theta' d\phi'}{\left(z^2+r'^2-2zr'\cos\theta'\right)^{\frac{1}{2}}}$$

$$\mu=\cos\theta'$$

$$V_{Gravity}=-G\rho 2\pi\int_0^{R_{Sun}}r'^2dr'\int_{-1}^{+1}\frac{d\mu}{\left(z^2+r'^2-2zr'\mu\right)^{\frac{1}{2}}}=G\rho 2\pi\int_0^{R_{Sun}}r'^2dr'\left.\frac{\left(z^2+r'^2-2zr'\mu\right)^{\frac{1}{2}}}{zr'}\right|_{-1}^{+1}$$

$$V_{Gravity}=G\rho 2\pi\int_0^{R_{Sun}}r'^2dr'\frac{\left[(z+r')-(r'-z)\right]}{zr'}=G\rho 2\pi\left(\int_0^{z}r'^2dr'\frac{2}{z}+\int_z^{R_{Sun}}r'^2dr'\frac{2}{r'}\right)$$

$$V_{Gravity}=G\rho 4\pi\left(\frac{1}{z}\int_0^{z}r'^2dr'+\int_z^{R_{Sun}}r'dr'\right)=G\rho 4\pi\left(\frac{z^3}{3z}+\frac{R_{Sun}^2}{2}-\frac{z^2}{2}\right)=G\rho 4\pi\left(\frac{2z^2}{6}+\frac{3R_{Sun}^2}{6}-\frac{3z^2}{6}\right)$$

$$V_{Gravity}=G\rho\frac{4\pi}{6}\left(3R_{Sun}^2-r^2\right)\qquad r<R_{Sun}$$

$$V_{Gravity}=-G\rho\frac{4\pi}{6}\left(3R_{Sun}^2-r^2\right)\qquad r<R_{Sun}$$

$$V_{Gravity}=-G\frac{\rho\frac{4}{3}\pi R_{Sun}^3}{r}=-G\frac{M_{Sun}}{r}\qquad r>R_{Sun}$$

$$U=\tfrac{1}{2}\int_{allspace}\rho V_{Gravity}d\tau=-\int_0^{2\pi}\int_0^{\pi}\int_0^{R_{Sun}}G\rho\rho\frac{4\pi}{6}\left(3R_{Sun}^2-r^2\right)d\tau$$

$$U=-\tfrac{1}{2}G\rho^2\frac{2\pi}{3}\int_0^{2\pi}\int_0^{\pi}\int_0^{R_{Sun}}\left(3R_{Sun}^2-r^2\right)r^2dr\sin\theta d\theta d\phi$$

$$U=-\tfrac{1}{2}G\rho^2\frac{2\pi}{3}4\pi\left[\int_0^{R_{Sun}}\left(3R_{Sun}^2r^2\right)dr-\int_0^{R_{Sun}}r^4dr\right]$$

$$U=-\tfrac{1}{2}G\rho^2\frac{2\pi}{3}4\pi\left[\frac{3R_{Sun}^5}{3}-\frac{R_{Sun}^5}{5}\right]$$

$$U=-\tfrac{1}{2}G\rho^2\frac{2\pi}{3}4\pi R_{Sun}^5\left[\frac{4R_{Sun}^5}{5}\right]$$

$$U=-\tfrac{1}{2}G\left(\rho\frac{4\pi R_{Sun}^3}{3}\right)^2\left[\frac{6}{5R_{Sun}}\right]$$

$$U=-\frac{3GM_{Sun}^2}{5R_{Sun}}$$

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$$U = -\frac{3GM_{Sun}^2}{5R_{Sun}} = -\frac{3\left(6.67\cdot 10^{-8}\,\text{dyne cm g}^{-2}\right)\left(2\cdot 10^{33}\,\text{g}\right)^2}{5\left(7\cdot 10^{10}\,\text{cm}\right)}$$

$$U = -2.3\cdot 10^{48}\,\text{dyne} = -2.3\cdot 10^{41}\,\text{J}$$

b) Assume that the total thermal kinetic energy of the atoms in the Sun is equal to -1/2 times gravitational energy. This is the result of the virial theorem in mechanics. Estimate the average temperature of the Sun. Take the number of particles as $1\text{x}10^{57}$. This estimate gives somewhat too low a temperature because the density of the Sun is far from uniform.

$$U_{Thermal} = -\tfrac{1}{2}U_{Gravity} = 1.1\cdot 10^{48}\,\text{dyne} = 1.1\cdot 10^{41}\,\text{J}$$

$$U_{Thermal} = \tfrac{3}{2}Nk_BT$$

$$T = \frac{2U_{Thermal}}{3Nk_B} = \frac{2\left(1.1\cdot 10^{41}\,\text{J}\right)}{3\left(1\cdot 10^{57}\,\text{particles}\right)\left(1.38\cdot 10^{-23}\,\text{JK}^{-1}\right)} = 5.3\cdot 10^{6}\,K$$

5. Surface temperature of the Earth

Calculate the temperature of the surface of the Earth on the assumption that as a black body in thermal equilibrium it reradiates as much thermal radiation as it receives from the Sun. Assume also that the surface of the Earth is at a constant temperature over the day-night cycle.

$$U_{in} = \sigma_B T_{Sun}^4 \frac{4\pi R_{Sun}^2}{4\pi R_{EarthToSun}^2}$$

$$P_{in} = U_{in}\pi R_{Earth}^2 = \sigma_B T_{Sun}^4 \frac{4\pi R_{Sun}^2}{4\pi R_{EarthToSun}^2}\pi R_{Earth}^2$$

$$P_{out} = P_{in} = \sigma_B T_{Earth}^4 4\pi R_{Earth}^2 = \sigma_B T_{Sun}^4 \frac{4\pi R_{Sun}^2}{4\pi R_{EarthToSun}^2}\pi R_{Earth}^2$$

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$$T_{Earth}^4 = T_{Sun}^4 \frac{R_{Sun}^2}{4R_{EarthToSun}^2}$$

$$T_{Earth} = T_{Sun}\left(\frac{R_{Sun}}{2R_{EarthToSun}}\right)^{\frac{1}{2}} = 5800K\left(\frac{7\cdot 10^{10}\,cm}{2\left(1.5\cdot 10^{13}\,cm\right)}\right)^{\frac{1}{2}} = 280K$$

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10.2 Assume the Sun emits radiation with the properties of a black-body at 6000K. What is the power radiated by the Sun per megacycle bandwidth at a wavelength of 2 cm? Radius of Sun is 7x10^{-10} cm.

Planck's radiation law for the radiation energy per unit volume per unit range of frequency:

$$u_\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/kT} - 1}$$

$$\omega = \frac{2\pi c}{\lambda} = \frac{2\pi 3 \cdot 10^8 m/s}{0.02m} = 9.4 \cdot 10^{10} s^{-1}$$

Radiant energy:

$$U = u_\omega d\omega V_{Sun}$$

$$U = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/kT} - 1} d\omega \frac{4\pi R_{Sun}^3}{3}$$

$$U = \frac{1.05459 \cdot 10^{-34} Js}{\pi^2 \left(3 \cdot 10^8 m/s\right)^3} \frac{\left(9.4 \cdot 10^{10} s^{-1}\right)^3}{e^{1.05459 \cdot 10^{-34} Js \left(9.4 \cdot 10^{10} s^{-1}\right) / \left(1.38 \cdot 10^{-23} JK^{-1}\right) 6000 K} - 1} 10^6 s^{-1} \frac{4\pi \left(7 \cdot 10^{12} m\right)^3}{3}$$

$$U = 8 \cdot 10^9 W$$

9.1 Classify the following particles according to the statistic which they obey:

i) 12C = BE

ii) 12C+ = FD

iii) 4He+ = FD

iv) H- = FD

v) 13C = ~~BE~~ FD

vi) positronium = BE

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7. Free Energy of a photon gas. (a) show that the partition function of a photon gas is given by:

$$Z = \prod_n \left[1 - e^{-\hbar\omega_n/kT}\right]^{-1}$$ Where the product is over the modes n.

Photons have spin 1 and obey bose-einstein statistics, and are non-interacting. The number of photons is not a constant (because they are emitted and absorbed during thermal equilibrium processes).

$$Z_{photon}(T,V) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} e^{-\sum_r n_r \varepsilon_r / kT}$$

$$X = \sum_{n1}^{\infty}\sum_{n2}^{\infty} e^{(a1n1+a2n2)} = \sum_{n1}^{\infty}\sum_{n2}^{\infty} e^{(a1n1)}e^{(a2n2)} = \sum_{n1}^{\infty} e^{(a1n1)} \sum_{n2}^{\infty} e^{(a2n2)} = \prod_{r=1}^{2} e^{(arnr)}$$

generalize

$$X = \sum_{n1}^{\infty}\sum_{n2}^{\infty} \dots e^{(a1n1+a2n2\dots)} = \prod_{nr=1}^{\infty} e^{(arnr)}$$

$$\sum_{nr=1}^{\infty} e^{(arnr)} = \frac{1}{1-e^{ar}}$$

let

$$a_r = \frac{-\varepsilon_r}{kT} = \frac{-\hbar\omega_r}{kT}$$

$$Z_{photon}(T,V) = \prod_n \left[1 - e^{-\hbar\omega_n/kT}\right]^{-1}$$

b) The Helmholtz free energy is found directly from this result as:

$$F = kT\sum_n^{\infty} \ln\left[1 - e^{-\hbar\omega_n/kT}\right]$$

$$f(\omega)d\omega = \frac{V\omega^2 d\omega}{\pi^2 c^3}$$

$$F = kT\int_0^{\infty} \ln\left[1 - e^{-\hbar\omega/kT}\right]\frac{V\omega^2 d\omega}{\pi^2 c^3}$$

$$x = \frac{\hbar\omega}{kT}$$

$$F = \frac{V(kT)^4}{\pi^2\hbar^3 c^3}\int_0^{\infty} \ln\left[1-e^{-x}\right]x^2 dx$$

$$\int_0^{\infty} \ln\left[1-e^{-x}\right]x^2 dx = \tfrac{1}{3}\int_0^{\infty} \ln\left[1-e^{-x}\right]dx^3 = \tfrac{1}{3}\ln\left[1-e^{-x}\right]x^3\Big|_0^{\infty} - \tfrac{1}{3}\int_0^{\infty}\frac{x^3}{e^x-1}dx = -\frac{\pi^2}{45}$$

$$F = -\frac{\pi^2 V(kT)^4}{45\hbar^3 c^3}$$

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