Last Lecture Ideal Bose Gas December 2008

Know how to derive:
- Ideal Bose gas
- Ideal Fermi gas
- Perfect Classical Gas

① Get $\bar{n}_i$

② In the classical limit $\bar{n}_i << 1$, bose + fermi look the same

③ $n = \sum n_i$

④ $E = \sum \bar{n}_i \varepsilon_i$

⑤ $\sum \to \int$

Don't follow Mandl, this discussion follows K&K

Ideal Bose Gas

$$\bar{n}_i = \frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1} \qquad \beta = \frac{1}{kT}$$

quantum state index i

as $i \uparrow$, $\varepsilon_i \uparrow$: $\varepsilon_0 \le \varepsilon_1 \le \varepsilon_2 \le \ldots \varepsilon_N$

usually chose $\varepsilon_0 = 0$

$$\bar{n}_0 = \frac{1}{e^{\beta(\varepsilon_0 - \mu)} - 1} = \frac{1}{e^{-\beta\mu} - 1} \quad \text{lowest state}$$

whenever $\bar{n}_0 > 0$ $\mu < 0$ for Bosons

$\lim T \to 0$ $\mu \to 0$ but it is negative

$$\frac{N}{V} = \frac{\bar{n}_0}{V} + \left(\frac{2\pi m k T}{h^2}\right)^{3/2} \frac{2}{\sqrt{\pi}} \underbrace{\int_0^\infty \frac{x^{1/2} z \, dx}{(e^x - z)}}_{\text{can integrate over } x = x(z)} = \frac{n_0}{V} + \left(\frac{2\pi m k T}{h^2}\right)^{3/2} g_{3/2}(z)$$

$$g_{3/2}(z) = \frac{x^{1/2} z}{e^x - z}$$

Integral $\int_0^\infty \frac{x^{1/2}\, z\, dx}{(e^x - z)}$

$z = e^{\beta\mu}$ $\quad \mu = -\infty \quad z = 0$ $\quad 0 \le z \le 1$

$\mu = 0 \quad z = 1$

↑ region of interest

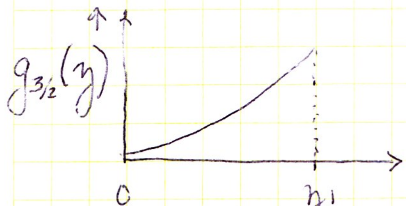
make an expansion, multiply by e^{-x}

$$\frac{x^{1/2} e^{-x} z\, dx}{1 - z e^{-x}}$$

Binomial series expansion - infinite series
integrate term-by-term

$\sum_{l=1}^{\infty} \frac{z^l}{l^{3/2}}$ highly convergent series

monotonically increasing function
whole function goes to zero
max value when $z = 1$
$\uparrow$ when $z \uparrow$



$g_{3/2}(z)$

$g_{3/2}(1) = \sum_{l=1}^{\infty} \frac{1}{l^{3/2}} \sim 2.612$ just keep the first few terms

$g_{3/2}(z) \le 2.612$

$$\frac{N}{V} = \frac{\bar{n}_0}{V} + \left(\frac{2\pi m k T}{h^2}\right)^{3/2} g_{3/2}(z)$$

When does B/E Condensation happens when significant fraction are in lowest state

$\frac{N}{V} \sim \frac{\bar{n}_0}{V} \qquad \frac{\bar{n}_0}{V} \ge 0$

$$\frac{\bar{n}_0}{V} = \frac{N}{V} - \left(\frac{2\pi m k T}{h^2}\right)^{3/2} g_{3/2}(z)$$

$\lambda_{DB} = \frac{h}{p} \qquad \bar{\lambda} = \frac{h}{\sqrt{\bar{p^2}}} \qquad \left(\frac{1}{\lambda^3}\right)$ quantum density
one molecule per λ_{DB}^3

average deBroglie wavelength

NOTE BOOK

Last Lecture T_c of BEC

For Bose-Einstein condensation

$$\frac{N}{V} > \left(\frac{2\pi mkT}{h^2}\right)^{3/2} g_{3/2}(z)$$

if $> g_{3/2}(1)$ condition is satisfied

and we have B/E Condensation

$$\frac{N}{V} > \left(\frac{2\pi mkT}{h^2}\right)^{3/2} (2.612)$$

$$\left(\frac{2\pi mkT}{h^2}\right)^{3/2} < \frac{N}{2.612\,V}$$

$$T \leq \left(\frac{h^2}{2\pi mk}\right)\left(\frac{1.0}{2.612}\frac{N}{V}\right)^{2/3}$$

condition for Bose-Einstein Condensation

$$T_c = \left(\frac{h^2}{2\pi mk}\right)\left(\frac{1}{2.612}\frac{N}{V}\right)^{2/3}$$

mass density

for most gases

$$T_c \sim \text{a few K}$$

this relation has been verified
doesn't remain a gas, it becomes a liquid
very dilute gas at low T
in 2001 MIT showed an alkali gas doesn't become a liquid or solid.

He_4 (alpha particle) is a Bose-Einstein Condensate { gas at STP, liquid at low T ~~or~~ 1 atm

He_3 is a Fermion

Consider liquid He_4 at 1atm ~~V =~~ $\frac{27.6\,cm^3}{mol} = V = \frac{1}{\text{molar density}}$

$$\frac{N}{V} = \frac{N_o}{V} = \frac{6.02\cdot 10^{23}\ \text{atoms/mol}}{27.6\ cm^3/mol}$$

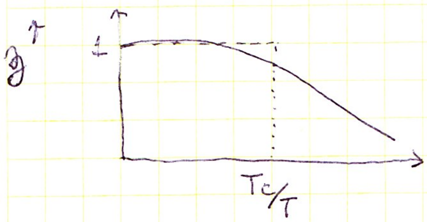
$T_c = 3.13\,K$ for liquid He_4 exhibits superfluidity!

He liquid can be roughly approximated by treating it as a gas,

$$\frac{N}{V} = \frac{\bar{n}_0}{V} + \frac{1}{\lambda_{DB}^3} g_{3/2}(z) = \frac{z}{V(1-z)} + \frac{1}{\lambda_{DB}^3} g_{3/2}(z)$$

can invert to get $z = z(N, V)$

$$\bar{n}_0 = \frac{1}{e^{-\beta\mu} - 1} = \frac{e^{\beta\mu}}{1 - e^{\beta\mu}} = \frac{z}{1-z}$$



For all practical purposes $T < T_c$ $z \simeq 1$ $\mu \simeq 0$ but $\mu < 0$

$$\frac{N}{V} = \frac{\bar{n}_0}{V} + \underbrace{\frac{1}{\lambda_{DB}^3} g_{3/2}(z)}_{\text{not in the lowest state}} = \frac{\bar{n}_0}{V} + \frac{N_1}{V}$$

$$\frac{N_1}{V} = \frac{1}{\lambda_{DB}^3} g_{3/2}(z)$$

for ~~the~~ $T < T_c$ $g_{3/2}(z) \sim g_{3/2}(1) = 2.612$

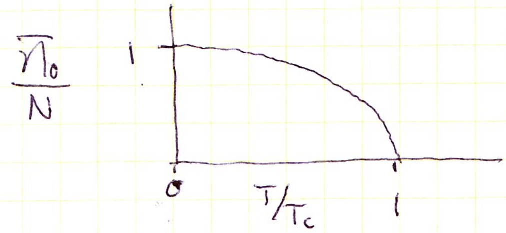
$$\frac{N_1}{V} = \frac{2.612}{\lambda_{DB}^3} = \left(\frac{2\pi m k T}{h^2}\right)^{3/2} (2.612)$$

$$\frac{\bar{n}_0}{V} \to 0 \quad \therefore \quad \frac{N}{V} \to \left(\frac{2\pi m k T}{h^2}\right)^{3/2} (2.612)$$

Ratio of these two $T < T_c$ $\frac{N_1}{N} = \left(\frac{T}{T_c}\right)^{3/2}$

$$\frac{\bar{n}_0}{N} = \frac{N - N_1}{N} = 1 - \frac{N_1}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$

Fraction of molecules in the lowest state when $T = 0$ $\bar{n}_0 = 1$
when $T = T_c$ $\bar{n}_0 = 0$



Last Lecture Internal Energy of BEC

Recall $\frac{V\,4\pi p^2 dp\,(2s+1)}{h^3}$ density of states and $E = \frac{p^2}{2m}$

Spin 0 $\frac{V\,4\pi p^2 dp}{h^3} = \frac{2\pi V(2mE)^{3/2}\,dE}{h^3}$

For $T < T_c$, what is the internal energy of the system?

$$\bar{E} = E = \sum_i \bar{\eta}_i \varepsilon_i = \bar{\eta}_0 \cdot 0 + \sum_{i=1}^{\infty} \bar{\eta}_i \varepsilon_i$$

$$\bar{E} = \int \frac{1}{e^{\beta(\mu-\varepsilon_i)} - 1}\, \varepsilon\, f(\varepsilon)\, d\varepsilon = \int \frac{1}{e^{\beta(\mu-\varepsilon)}-1}\, \varepsilon\, 2\pi V\left(\frac{2m}{h^2}\right)^{3/2} \varepsilon^{-1/2}\, d\varepsilon$$

$$\bar{E} = 2\pi V\left(\frac{2m}{h^2}\right)^{3/2} \int \frac{\varepsilon^{3/2}\, d\varepsilon}{e^{\beta(\mu-\varepsilon)} - 1}$$ want integrals to be dimensionless

$x = \beta\varepsilon \quad \varepsilon = (kT)x \quad d\varepsilon = (kT)dx \quad x$ is dimensionless variable

$$\bar{E} = 2\pi V\left(\frac{2m}{h^2}\right)^{3/2} \int \frac{(kT)^{3/2} x^{3/2}\,(kT)\,dx}{e^{-\beta(\mu-\varepsilon)} - 1} = 2\pi V\left(\frac{2m}{h^2}\right)^{3/2} (kT)^{5/2} \int_0^{\infty} \frac{x^{3/2}\, dx}{\frac{1}{z}e^{x} - 1}$$

$$\bar{E} = 2\pi V\left(\frac{2m}{h^2}\right)^{3/2} (kT)^{5/2} \int_0^{\infty} \frac{z\, x^{3/2}\, dx}{(e^{x} - z)}$$ expand this integrand and integrate term by term

$$\frac{x^{3/2} z}{e^{x} - z} = \frac{x^{3/2} z}{e^{x}(1 - z e^{-x})} = z x^{3/2} e^{-x}(1 - z e^{-x})^{-1}$$

since $z e^{-x} \leq 1$ can do a binomial series expansion

$$\frac{x^{3/2} z}{e^{x} - z} \sim z x^{3/2} e^{-x} \sum_{n=0}^{\infty} z^n e^{-nx} = x^{3/2} \sum_{l=1}^{\infty} z^l e^{-lx}$$

Substitute and do the integration

$$\bar{E} = 2\pi V\left(\frac{2m}{h^2}\right)^{3/2} (kT)^{5/2} \sum_{l=1}^{\infty} z^l \underbrace{\int_0^{\infty} x^{3/2} e^{-lx}\, dx}_{\text{gamma function when } y = lx \quad x = \frac{y}{l} \quad dx = \frac{dy}{l}}$$

$$\bar{E} = 2\pi V\left(\frac{2m}{h^2}\right)^{3/2} (kT)^{5/2} \sum_{l=1}^{\infty} \frac{z^l}{l^{5/2}} \underbrace{\int_0^{\infty} y^{3/2} e^{-y}\, dy}_{\Gamma(5/2) = \frac{3}{2}\Gamma(3/2) = (3/2)(1/2)\Gamma(1/2) = \frac{3}{4}\Gamma(\frac{1}{2}) = \frac{3}{4}\sqrt{\pi}} = 2\pi V\left(\frac{2m}{h^2} kT\right)^{3/2} kT\, \frac{3\sqrt{\pi}}{4} \sum_{l=1}^{\infty} \frac{z^l}{l^{5/2}}$$

NOTE BOOK

Internal Energy, C_V of BEC When $T \le T_c$

$$\bar{E} = 2\pi V \left(\frac{2mkT}{h^2}\right)^{3/2} kT \, \frac{3}{4}\sqrt{\pi} \sum_{l=1}^{\infty} \frac{z^l}{l^{5/2}} = \frac{3}{2} V (kT) \underbrace{\left(\frac{2\pi m k T}{h^2}\right)^{3/2}}_{1/\lambda_{DB}^3} \underbrace{\sum_{l=1}^{\infty} \frac{z^l}{l^{5/2}}}_{\text{convergent series}}$$

$\boxed{\text{for } T < T_c}$ $z \sim 1$ $\displaystyle\sum_{l=1}^{\infty} \frac{z^l}{l^{5/2}} = \sum_{l=1}^{\infty} l^{-5/2} \to 1.341$

$$\bar{E} = \frac{3}{2} V (kT) \left(\frac{2\pi kTm}{h^2}\right)^{3/2} (1.341)$$

$$\frac{E}{V} = \left(\frac{3}{2}\right)(1.341)(kT)\left(\frac{2\pi mkT}{h^2}\right)^{3/2}$$

previously $\displaystyle \frac{N}{V} = \left(\frac{2\pi mkT}{h^2}\right)^{3/2} 2.612$

$$\frac{\bar{E}}{N} = \frac{2.01}{2.612} kT \left(\frac{T}{T_c}\right)^{3/2} = .77\, k \frac{T^{5/2}}{T_c^{3/2}}$$

T_c depends on N and V!

$F = \bar{E} - TS$

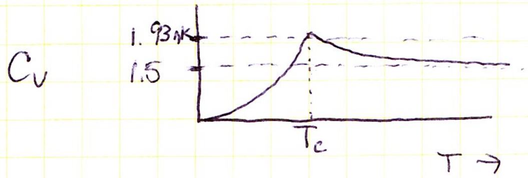
S from C_V

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{V,N} = 0.77\, Nk \, \frac{5}{2} \frac{T^{3/2}}{T_c^{3/2}} \sim 1.93\, Nk \left(\frac{T}{T_c}\right)^{3/2}$$

remember this is valid only when $T \le T_c$

$C_V \to 0$ when $T \to 0$ is required by the third law of thermodynamics

$C_V \to 1.93\, Nk$ when $T \to T_c$ very rapidly



at T_c there is a very sharp change in T_c
perfect gas $3/2\, nk$

$S = \int C_V \, dT$

these results are confirmed for BEC
time to stop...