Penultimate Lecture   Ideal Fermi Gas $T << T_F$

Ideal Fermi Gas $\quad T_F = \frac{\varepsilon_F}{k} >> T_{room}$

$$N = \Sigma \eta_i = \int_0^\infty n(\varepsilon) f(\varepsilon) d\varepsilon$$

$$\frac{4\pi p^2 dp}{h^3} \underset{\overset{\uparrow}{\text{spin}\,1/2}}{(2)} = \frac{8\pi V}{h^3} p^2 dp = \frac{8\pi V}{h^3} 2m\varepsilon \frac{\sqrt{2m\varepsilon}}{2} d\varepsilon = 4\pi V\left(\frac{2m}{h^2}\right)^{5/2} \varepsilon^{1/2} d\varepsilon$$

Number of Fermions in ideal Fermi Gas

$$N = 4\pi V \left(\frac{2m}{h^2}\right)^{3/2} \int_0^\infty \frac{\varepsilon^{1/2} d\varepsilon}{e^{\beta(\varepsilon-\mu)} + 1}$$

$N = N(V, \mu, T)$ can invert the relation to get $\mu = \mu(N, V, T)$

Internal energy of ideal Fermi Gas

$$\overline{E} = \sum_i \overline{\eta}_i \varepsilon_i = 4\pi V \left(\frac{2m}{h^2}\right)^{3/2} \int_0^\infty \frac{\varepsilon^{3/2} d\varepsilon}{e^{\beta(\varepsilon-\mu)} + 1} \qquad \overline{E} = \overline{E}(\mu, T, V)$$

When $T << T_F$ the formula can be easily derived

$$I = \int_0^\infty \frac{\phi(\varepsilon) d\varepsilon}{e^{\beta(\varepsilon-\mu)} + 1} \qquad \phi(\varepsilon) = \varepsilon^x$$

$$\boxed{I \approx \int_0^\mu \phi(\varepsilon) d\varepsilon + \frac{\pi^2}{6}(kT)^2 \frac{d\Phi(\mu)}{d\mu}}$$

will give copy of proof in folder...

Immediately write the expression

$$N = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\left[\int_0^\mu \varepsilon^{1/2} d\varepsilon + \frac{\pi^2}{6}(kT)^2\left(\frac{1}{2}\mu^{-1/2}\right)\right]$$

$$N = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\left[\frac{\mu^{3/2}}{3/2} + \frac{\pi^2}{6}(kT)^2 \frac{1}{2}\mu^{-1/2}\right]$$ = constant independent of temperature

$$N = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\frac{\varepsilon_F^{3/2}}{(3/2)} \longrightarrow 4\pi V\left(\frac{2m}{h^2}\right)^{3/2} = \frac{3N}{2\varepsilon_F^{3/2}}$$

$$\cancel{N} = \frac{3\cancel{N}}{2\varepsilon_F^{3/2}}\left[\frac{\mu^{3/2}}{3/2} + \frac{\pi^2}{6}(kT)^2\frac{1}{2\mu^{1/2}}\right]$$

Ideal Fermi Gas

$$1 = \frac{\mu^{3/2}}{\varepsilon_F^{3/2}}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right] \rightarrow \cancel{\frac{\mu^{3/2}}{\varepsilon_F}}$$

Solve for $\mu = \mu(\varepsilon_F, T)$ assume $\left(\frac{kT}{\mu}\right)^2 \ll 1$

then replace μ with ε_F in that denominator

$$\left(\frac{kT}{\mu}\right)^2 \rightarrow \left(\frac{T}{T_F}\right)^2$$

$$\mu^{3/2} \simeq \varepsilon_F^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right]^{-1} \sim \varepsilon_F^{3/2}\left[1 - \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right] \sim \varepsilon_F^{3/2}\left[1 - \frac{\pi^2}{8}\left(\frac{T}{T_F}\right)^2\right]$$

$$\mu \sim \varepsilon_F\left[1 + \frac{\pi^2}{8}\left(\frac{T}{T_F}\right)^2\right]^{2/3}$$

$$\mu \sim \varepsilon_F\left[1 - \frac{2}{3}\left(\frac{\pi^2}{8}\right)\left(\frac{T}{T_F}\right)^2\right] \quad \text{for } T \ll T_F$$

$$\mu \sim \varepsilon_F\left(1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right)$$ our assumption that ε_F is very close to ε_F even at room temperature

Get the energy of the conduction electrons in ~~th~~ terms of T then can get the heat capacity as a function of temperature

$$\bar{E} = \frac{3N}{2\varepsilon_F^{3/2}}\left[\int_0^{\mu} \varepsilon^{3/2} d\varepsilon + \frac{\pi^2}{6}(kT)^2\left(\frac{3\mu^{1/2}}{2}\right)\right] = \frac{3}{5}N\frac{\mu^{5/2}}{\varepsilon_F^{3/2}}\left[1 + \frac{5\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right]$$

again replace μ with ε_F with negligable error

$$\bar{E} \sim \frac{3}{5}N\frac{\mu^{5/2}}{\varepsilon^{3/2}}\left[1 + \frac{5}{8}\pi^2\left(\frac{T}{T_F}\right)^2\right] = \frac{3}{5}N\frac{\left(\varepsilon_F^{5/2}\left(1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right)^{5/2}\right)}{\varepsilon_F^{3/2}}\left[1 + \frac{5\pi^2}{8}\left(\frac{T}{T_F}\right)^2\right]$$

$$\bar{E} = \frac{3}{5}N\varepsilon_F\left(1 - \frac{5}{24}\pi^2\left(\frac{T}{T_F}\right)^2\right)\left(1 + \frac{5\pi^2}{8}\left(\frac{T}{T_F}\right)^2\right)$$

Product of brackets, neglect the 4th order T/T_F terms

$$\bar{E} = \underbrace{\frac{3}{5}N\varepsilon_F}\left(1 + \frac{5}{12}\pi^2\left(\frac{T}{T_F}\right)^2\right)$$

most of the energy is $\frac{3}{5}N\varepsilon_F$, the energy at absolute zero.

$$C_V = \frac{\partial E}{\partial T} = \frac{\pi^2}{2}\varepsilon_F\frac{T}{T_F^2} = \frac{\pi^2 NkT}{2T_F}$$

$C_V \propto T$ up to 1000K

valid at Troom $T \ll T_F$

$T_F \sim 25{,}000\text{K}$ to $70{,}000\text{K}$

Penultimate Lecture Ideal Boson Gas

Vibration dominates over conduction electron terms
derived the heat capacity due to conduction electron
true for any ideal Fermi gas

Entropy $\Delta S = \int \frac{dQ}{T} \Rightarrow S(T) - S(T=0K) = \int_0^T \frac{C_v\,dT}{T} = \frac{\pi^2}{2}\frac{Nk}{T_F}\int_0^T dT$

$\Delta S = \left(\frac{\pi^2}{2}\frac{Nk}{T_F}\right)T$ looks exactly like the expression for C_v in this case

Blackbody radiation $E_{BB} \propto T^4$ $E \propto T^3$ valid at all T

$$C_v \propto T^3$$

$S(T) = \int_0^T C_v \frac{dT}{T}$ entropy due to BB radiation due to T^3

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end Fermi Gas

Ideal Boson Gas

No Pauli exclusion principle at ↑T all behave like a perfect classical gas. But at $T \to 0$, all the Bosons all crowd into lowest possible quantum state

$$N = \sum_i \overline{n}_i$$

$\overline{n}_i$ can be any number $\begin{cases} n_i \ll 1 & \text{for classical} \\ 0 < n_i \leq 1 & \text{for ideal Fermi gas} \\ n_i \text{ any} & \text{value for ideal Bose gas} \end{cases}$

Suppose a lot of particles are in the lowest state

$$N = \sum_i \overline{n}_i = \overline{n}_0 + \overline{n}_1 + \cdots + \int_{\underset{\nwarrow ?}{\varepsilon_0}}^{\infty} \frac{1}{e^{\beta(\varepsilon - \mu)} - 1} f(\varepsilon)\,d\varepsilon$$

$\overline{n}_i = \frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1}$ i labels the quantum state

$\overline{n}_0 = \frac{1}{e^{-\mu\beta} - 1}$ let $\varepsilon_0 = 0$

Suppose $\mu > 0$ then $\overline{n}_0 = \frac{1}{e^{-|x|} - 1} \rightarrow \overline{n}_0 < 0$

But the number of Bosons in the lowest state can't be zero $\therefore \mu < 0$

From this result $\mu > 0$ and in general $\mu < \varepsilon_0$

Ideal Boson Gas

Thus:

The chemical potential is: $\mu < 0$ Bosons
$\mu > 0$ Fermions

$$\overline{n}_0 = \frac{1}{e^{-\mu/kT} - 1}$$

Suppose $T \to 0$ and $\mu < 0$ and $|\mu| \to 0$

Then you get a situation that $\overline{n}_0$ becomes quite large - close to fraction of N_0 - then you get a Bose-Einstein condensation

$$N = \overline{n}_0 + \overline{n}_1 + \dots + \dots$$

Boson has spin 0 $\therefore$ $f(\varepsilon)d\varepsilon = \frac{4\pi V p^2 dp}{h^3}(2\overset{1}{\cancel{s+1}})$ $\quad \varepsilon = \frac{p^2}{2m}$

$$= 2V\pi \left(\frac{2m}{h^2}\right)^{3/2} \varepsilon^{1/2} d\varepsilon$$

$$N\ \overline{n}_i = 2\pi V\left(\frac{2m}{h^2}\right)^{3/2} \int_{\varepsilon_{min}}^{\infty} \frac{\varepsilon^{1/2} d\varepsilon}{(e^{-\beta\mu} e^{\beta\varepsilon} - 1)}$$

$$\frac{N}{V} = \frac{\overline{n}_0}{V} + \frac{\overline{n}_1}{V} + \dots + = 2\pi\left(\frac{2m}{h^2}\right)^{3/2} \int_{\varepsilon_{min}}^{\infty} \frac{\varepsilon^{1/2} d\varepsilon}{(e^{-\beta\mu} e^{\beta\varepsilon} - 1)}$$

can replace ε_{min} with 0

$$\frac{N}{V} = \frac{\overline{n}_0}{V} + 2\pi\left(\frac{2m}{h^2}\right)^{3/2} \int_0^{\infty} \frac{\varepsilon^{1/2} d\varepsilon}{(e^{-\beta\mu} e^{\beta\varepsilon} - 1)}$$

$z = e^{\beta\mu} = e^{\mu/kT}$ $\therefore$ $0 < z \leq 1$ let $x = \beta\varepsilon$ a dimensionless variable

$$\frac{N}{V} = \frac{\overline{n}_0}{V} + 2\pi\left(\frac{2m}{h^2}\right)^{3/2}(kT)^{3/2} \underbrace{\int_0^{\infty} \frac{x^{1/2} z\, dx}{e^x - z}}$$

numerically solve - or Taylor series expand integral & integrate term by term.

$$\frac{z x^{1/2}}{(e^x - z)} = \frac{z e^{-x} x^{1/2}}{(1 - z e^{-x})} = z x^{1/2} e^{-x} \sum_{n=0}^{\infty} z^n e^{-nx} = x^{1/2} \sum_{\ell=1}^{\infty} z^\ell e^{-\ell x} \to \Gamma \text{ function}$$

← can see it is <1 so binomial exp.

$$\frac{N}{V} = \frac{\overline{n}_0}{V} + 2\pi\left(\frac{2mkT}{h^2}\right)^{3/2} \sum_{\ell=1}^{\infty} z^\ell \int_0^{\infty} x^{1/2} e^{-\ell x} dx = \frac{n_0}{V} + 2\pi\left(\frac{2mkT}{h^2}\right)^{3/2} \sum_{\ell=1}^{\infty} z^\ell \underbrace{\int_0^{\infty} \frac{y^{1/2} e^{-y} dy}{\ell^{3/2}}}_{\sqrt{\pi}/2}$$

$$\frac{N}{V} = \frac{\overline{n}_0}{V} + 2\pi\left(\frac{2mkT}{h^2}\right)^{3/2} \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell^{3/2}} \frac{\sqrt{\pi}}{2} \ *$$

$\ell x = y$
$\ell dx = dy$

* highly convergent series