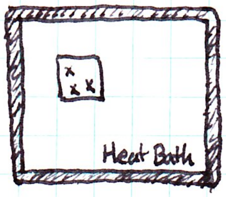
Lecture — Ideal Fermi Gas at Any Temperature



We want to let the number of particles to vary.

Each single particle is in a particular state where ε_i is the total energy of the state.

$$E_{TOT} = \sum_i n_i \varepsilon_i$$

$$Z = \sum_{n_1 n_2 n_3 \dots} e^{-\beta(E-\mu N)} = \sum_{n_1 n_2 \dots} e^{-\beta \sum_i n_i(\varepsilon_i - \mu)}$$

$$Z = \sum_{n_1}\sum_{n_2}\dots\sum_{n_N} e^{-\beta[n_1(\varepsilon_1-\mu) + n_2(\varepsilon_2-\mu) + \dots n_N(\varepsilon_N-\mu)]}$$

$$Z = \underbrace{\sum_{n_1} e^{-\beta n_1(\varepsilon_1-\mu)}}_{z_1} \underbrace{\sum_{n_2} e^{-\beta n_2(\varepsilon_2-\mu)}}_{z_2} \dots \underbrace{\sum_{n_N} e^{-\beta n_N(\varepsilon_N-\mu)}}_{z_N} = z_1 z_2 \dots z_N$$

$$\Rightarrow \quad Z = \prod_i^{\infty} z_i$$

Single $i^{\underline{th}}$ particle $\quad z_{iF} = \sum_{n_i} e^{-\beta n_i(\varepsilon_i - \mu)}$

From the integral $\sum_i \to \int$

Fermions	Bosons
$z_{iF} = 1 + e^{-\beta(\varepsilon_i-\mu)}$	$z_{iB} = 1 - e^{-\beta(\varepsilon_i-\mu)}$

in order for this to hold $\varepsilon_i - \mu > 0 \; \therefore \; \varepsilon_i > \mu$

$$\varepsilon_i > \varepsilon_{i-1} > \dots > \varepsilon_2 > \varepsilon_1 > \mu \quad \therefore \; \mu < 0$$

and μ is the lowest energy level

$$P(n_i) = \frac{e^{-\beta n_i(\varepsilon_i-\mu)}}{Z}$$ = (how many particles are in that state $= n_i$ / probability of finding n_i particles

$$P(\varepsilon_i) = \frac{e^{-\beta(\varepsilon_i-\mu)}}{Z}$$ = (probability of finding ε_i energy

Probability of finding 3 particles in the 1st excited state:

$$P(3_1) = \frac{e^{-\beta 3_1(\varepsilon_1 - \mu)}}{\mathcal{Z}_1} = P(E = 3\varepsilon_i)$$

$$(a+b+c+\cdots)\frac{\sum\sum\sum\cdots}{Z} = 1 = \frac{a\sum_{n_2}\sum_{n_3}\sum_{n_4}\cdots + b\sum\sum\sum}{\sum_{n_1}\sum_{n_2}\sum_{n_3}\cdots} = \frac{a}{Z_1}$$

expand $\sum_{n_i}$

In general for any $\mathcal{Z}_i$

What is the expected number of particles in the i^{th} state

$$\overline{n_i} = \sum n_i P(n_i) = \frac{\sum n_i e^{-\beta n_i(\varepsilon_i - \mu)}}{Z} = \sum \frac{n_i e^{-\beta n_i(\varepsilon_i-\mu)}}{\sum_{n_i} e^{-\beta n_i(\varepsilon_i-\mu)}} = kT\frac{\partial \ln(\mathcal{Z}_i)}{\partial \mu}$$

$$\overline{n_i} = kT\frac{(1-e^{-\beta(\varepsilon_i-\mu)})\beta}{(1-e^{-\beta(\varepsilon_i-\mu)})^2} = \frac{1}{e^{\beta(\varepsilon_i-\mu)} \pm 1}$$

$+$ = Fermions

$-$ = Bosons

Recall Conditions $\overline{n_i} << 1$, T interaction density δ

Bosons $\mu < 0$

Fermions $\mu > 0$

$M = (n_+ + n_-)B\mu$ $\quad \overline{n_i} = \frac{1}{e^{\beta(\varepsilon_i-\mu)}+1}$ $\quad \varepsilon_i < \mu$ $\quad$ as $T \to 0$ $n_i \to 1$

$\varepsilon_{max} = \varepsilon_F < \mu$

p. 285 have to go to the total number of particles

$$N = \int_0^\infty \overline{n_i} f(\varepsilon)d\varepsilon = A\int_0^\infty \frac{\varepsilon^{1/2}\, d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

want integral to converge $\quad$ contradiction: can't have negative # of particles

if $\mu < 0$ $\quad e^{-\beta\mu} \to \infty$ $\quad N \to 0$ $\quad \mu = \mu(T, V, N) \to \mu(T)$ $\quad \mu(T=0) = \varepsilon_F$

$\varepsilon_F > 0$ to avoid $N \to 0$ $\quad \varepsilon_i < \varepsilon_F$ for occupied state

$\varepsilon_i > \varepsilon_F$ for unoccupied state

$$\overline{n_i} = \frac{1}{e^{\beta(\varepsilon_i - \varepsilon_F)}+1} = \begin{cases} 0 & \varepsilon_i - \varepsilon_F > 0 \quad \varepsilon_i > \varepsilon_F \quad \text{unoccupied state} \\ 1 & \varepsilon_i - \varepsilon_F < 0 \quad \varepsilon_i < \varepsilon_F \quad \text{occupied state} \end{cases}$$

interaction energy $\quad \overline{n} = \overline{n}(\varepsilon \mp \mu_B B)$ $\quad \varepsilon_i = \varepsilon_{ti} \mp \mu B$ $\quad \varepsilon_i = \varepsilon_F \pm i\mu B$

$d\varepsilon_i = d\varepsilon_{ti}$

$$C_V = \frac{\partial E}{\partial T} = \frac{1.93R}{T_c^{3/2}}T^{3/2} \quad T < T_c \quad \therefore E = \frac{1.93R}{5/2\, T_c^{3/2}} T_{5/2} + \text{constant}$$

$$F = E - TS \quad \Delta S = \frac{\Delta Q}{T} \quad \Delta Q = NC_V\Delta T \quad S = A\int T^{1/2}dT = \frac{AT^{3/2}}{3/2} + \text{constant}$$

$$= \frac{E^{5/2}AT^{3/2}}{AT^{5/2}(3/2)} = \frac{5}{3}\frac{E}{T}$$

$$P = \frac{\partial S}{\partial V} \quad T_c = 11.1$$

$\overline{n}_i \leq 1$ Pauli exclusion principle - always only one fermion per quantum state

$$\overline{n}_i = \frac{1}{e^{\beta(\varepsilon_i - \mu)}+1} \qquad f(\varepsilon)d\varepsilon = \frac{V\,4\pi p^2 dp\,(2s+1)}{h^3} \qquad \varepsilon = \frac{p^2}{2m}$$

$$N = \sum_i \overline{n}_i = \int_0^\infty \frac{1}{e^{\beta(\varepsilon_i-\mu)}+1} f(\varepsilon)d\varepsilon = \int_0^\infty \frac{1}{e^{\beta(\varepsilon_i-\mu)}+1}\, 4\pi V \left(\frac{2m}{h^2}\right)^{3/2} \varepsilon^{1/2} d\varepsilon$$

$$x = \frac{\varepsilon}{kT} \qquad \varepsilon = kTx \qquad d\varepsilon = kT\,dx \qquad \beta\varepsilon = x$$

$$N = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2} \int_0^\infty \frac{(kT)^{3/2} x^{1/2} dx}{e^{-\beta\mu}e^{x}+1} = 4\pi V\left(\frac{2mkT}{h^2}\right)^{3/2}\int_0^\infty \frac{x^{1/2}dx}{e^{-\beta\mu}e^{x}+1}$$

First consider when $T \to 0$

$$\lim_{T\to 0} \beta\mu = \lim_{T\to 0}\frac{\mu}{kT} = \pm\infty$$ depends on whether μ is positive or negative

$$\varepsilon_F = \mu(T=0)$$

If $\varepsilon_F < 0$ then the integral vanishes as $T \to 0$

If $\varepsilon_F > 0$ then take limit $\overline{n}_i(T=0) = \dfrac{1}{e^{(\varepsilon_i - \varepsilon_F)/kT}+1} = \begin{cases} 1 & \text{if } \varepsilon_i < \varepsilon_F \\ 0 & \text{if } \varepsilon_i > \varepsilon_F \end{cases}$

At absolute 0, the fermions go to the lowest possible state. Even at absolute 0, the system has finite energy because $\overline{n}_i(T=0)=1$

$$N = \int_0^{\varepsilon_F} \overline{n}_i f(\varepsilon)d\varepsilon \Rightarrow \int_0^{\varepsilon_F} f(\varepsilon)d\varepsilon = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\int_0^{\varepsilon_F}\varepsilon^{1/2}d\varepsilon = \frac{8}{3}\pi V\left(\frac{2m\varepsilon_F}{h^2}\right)^{3/2} \text{ @ } T=0$$

Can invert this to get an expression for $\varepsilon_F = \varepsilon_F(N,V)$

$$\varepsilon_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}$$

$$\frac{N}{V} = \frac{\#\text{atoms}}{\text{volume}} = \frac{\#\text{conduction } e^-\text{'s}}{\text{Volume}} = \left(\frac{N_0}{A}\right)\rho$$

mass density

#atoms per gram

The table in the book gives you Fermi energy for different temperatures.

$$T_F = \frac{\varepsilon_F}{k}$$

Fermi Gas: Energy and Pressure at absolute zero

Can calculate the energy at absolute zero when $\varepsilon_F > 0$

$$\overline{E} = \sum_i \varepsilon_i \overline{n}_i = \int_0^{\varepsilon_F} \varepsilon f(\varepsilon)\,d\varepsilon = 4\pi V \left(\frac{2m}{h^2}\right)^{3/2} \int_0^{\varepsilon_F} \varepsilon^{3/2} d\varepsilon = 4\pi V \left(\frac{2m}{h^2}\right)^{3/2} \frac{\varepsilon_F^{5/2}}{5/2} = \frac{3N\varepsilon_F}{5}$$

$$\overline{E} = \frac{3N\varepsilon_F}{5} \qquad \varepsilon_F > 0$$

The average energy is somewhere between ε and ε_F

$$F = E - TS \qquad \lim_{T\to 0} F = E - \cancel{T}^{\,0} S = E \quad \text{since } S \text{ is finite}$$

Even at absolute zero a fermion gas will exert pressure on the walls of its container.

Substitute ε_F in terms of V

$$F = \overline{E} = \frac{3N\varepsilon_F}{5} = \frac{3N}{5}\left(\frac{h^2}{2m}\right)\left(\frac{3N}{8\pi V}\right)^{2/3} = \frac{(3N)^{5/3} h^2}{80\pi} V^{-2/3} \qquad \text{at } T = 0$$

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \frac{2}{3}\left(\frac{(3N)^{5/3} h^2}{80\pi}\right) V^{-5/3} = \frac{h^2}{120\pi}\left(\frac{3N}{V}\right)^{5/3} \qquad \text{at } T = 0$$