Lecture 21 Ideal Gas, Bosons & Fermions

An ideal gas is a collection of very, very weakly interacting particles. The particles can move anywhere in their container. If the container is a cube can solve the time-independent Schrodinger equation to get the quantum states:

$$E_{n1,n2,n3} = \frac{\hbar^2\pi^2}{2mL^2}\left(n_1^2 + n_2^2 + n_3^2\right)$$

$$\psi_{n1\,n2\,n3} = \left(\frac{2}{L}\right)^{3/2} \sin\left(\frac{n_1\pi x}{L}\right)\sin\left(\frac{n_2\pi y}{L}\right)\sin\left(\frac{n_3\pi z}{L}\right)$$

There will be some degeneracy

	n_1	n_2	n_3
DEGEN	1	0	0
	0	1	0
	0	0	1

etc. $\psi_{100}, \psi_{010}, \psi_{001}$ have same energy

Some quantum states for each particle: all we have to know is each particle has a set of quantum states, $1, 2, 3, \ldots, i, \ldots$ all identical particles is ... eg If Hydrogen gas, every particle is Hydrogen by we don't know which particle is in which state. The quantum states keep changing. Chemists call the quantum states: orbitals.

n_i = the number of particles in the i^{th} quantum state

The grand canonical partition function gives $\bar{n}_i$, from that you can get $\bar{E}$, P, etc. of the gas - all are statistic averages.

System = The i^{th} quantum state (orbital)
heat bath = the rest of the quantum states

$$\mathcal{Z}_i = \sum_{N_r} e^{\beta(\mu N - E_{N_r})}$$

This sum depends on whether the particles are bosons, all particles can be in any quantum state, or fermions, only one particle can exist in a given quantum state.

Fermion	Antisymmetric Wavefunction	Only 1 per quantum state
Boson	Symmetric Wavefunction	

Fermions - Can have 0 or 1 in the i^{th} quantum state

$$\mathcal{Z}_i = \underbrace{1}_{\text{zero}} + \underbrace{e^{\beta(\mu - E_{N_r})}}_{\text{1 in that state}} = 1 + e^{\beta(\mu - E_i)}$$

$$\bar{n}_i = kT\frac{\partial \ln \mathcal{Z}_i}{\partial \mu} = \frac{kT}{\mathcal{Z}_i}\beta e^{\beta(\mu - E_i)} = \frac{e^{\beta(\mu - E_i)}}{\left[1 + e^{\beta(\mu - E_i)}\right]}$$

$$\boxed{\text{Fermions: } \quad \bar{n}_i = \frac{1}{(1+e^{\beta(\varepsilon_i-\mu)})} = \text{either } \emptyset \text{ or } 1}$$

Bosons: No Pauli exclusion principle.

$$\mathcal{Z}_i = \sum_{N_r} e^{\beta(\mu N - E_r)} = 1 + e^{\beta(\mu-\varepsilon_i)} + e^{2\beta(\mu-\varepsilon_i)} + \dots + e^{N\beta(\mu-\varepsilon_i)}$$

converging series, Binomial expansion $\frac{1}{1-x}$

assume $\varepsilon_i > \mu$ for all i

$$\mathcal{Z}_i = \frac{1}{1-e^{\beta(\mu-\varepsilon_i)}}$$

$$\bar{n}_i = kT\frac{\partial \ln \mathcal{Z}_i}{\partial \mu} = kT\frac{1}{\mathcal{Z}_i}\frac{\partial \mathcal{Z}_i}{\partial \mu} = kT(1-e^{\beta(\mu-\varepsilon_i)})\frac{e^{\beta(\mu-\varepsilon_i)}}{(1-e^{\beta(\mu-\varepsilon_i)})}$$

$$\boxed{\text{Bosons: } \quad \bar{n}_i = \frac{1}{e^{\beta(\varepsilon_i-\mu)}-1}}$$

Comparing Expressions

Fermion	$0 \le \bar{n}_i \le 1$	$\dfrac{1}{e^{\beta(\varepsilon_i-\mu)}+1}$
Boson	$0 < \bar{n}_i$	$\dfrac{1}{e^{\beta(\varepsilon_i-\mu)}-1}$

At high T, P, all ideal gases behave as $PV = RT = NkT$

How can this be true if there are both Fermions + Bosons?

Suppose $\bar{n}_i \ll 1$ for all i (this is the classical condition)

For He gas, this condition is easily satisfied

But this condition is never satisfied at e^-'s in conduction band, or low temperature gas, He^4 (Boson) vs. He^3 (Fermion)