# Lecture 5 Paramagnetic Solids in a Magnetic Field

$\vec{B}=0 \rightarrow \vec{M}_{net}=0$ for a paramagnetic solid

But when $\vec{B}\neq 0$, $\mu$ orient by $\vec{\tau}=\vec{\mu}\times\vec{B}$ in direction of field.
The combination of thermal agitation causes less orientation
How can we find $\vec{M}$ ?

Consider as a system in equilibrium with a heat bath with $\vec{B}=+B_z\hat{z}$

| | Classical | Quantum |
|---|---|---|
| Interaction Energy $E$ | $E=-\vec{\mu}\cdot\vec{B}=-\mu_z B$ | $E=-\vec{\mu}\cdot\vec{B}=-\mu_z B$ |
| Angular momentum | $\vec{\mu}\propto\vec{J}\rightarrow\mu_z\propto J_z$ | $\vec{\mu}\propto\vec{J}\rightarrow\mu_z\propto J_z$ |
| $\mu$ of $\vec{J}$, spin only | $J_z$ can be any value | $J_z$ is quantized in multiples of $\hbar$<br>$\{+\hbar j, \hbar(j-1), \ldots, -\hbar j\}$<br>with $2j+1$ different values |
| Gyromagnetic Ratio $g$ | $g=1$ | $g=$ a number (eigenvalue)<br>$=1, 2, 2.8, \ldots$ |
| Magnetic moment $\vec{\mu}$ | $\mu_z=\frac{e}{2mc}\,g\,J_z$ | $\mu_z=\frac{e}{2mc}\,g\,J_z$ |

if $j=0$, $\mu_z=0$ $\mu_z=J_z=1$ spin 1

suppose $J_z=\{+\hbar/2, -\hbar/2\}$

$$\mu_z=\left\{\frac{e}{2mc}g\left(\frac{\hbar}{2}\right), \frac{-e}{2mc}g\left(\frac{\hbar}{2}\right)\right\}$$
$$=\{+\mu, -\mu\}$$

| | Classical | Quantum |
|---|---|---|
| Energy of Magnetic Moment | $-\mu_z B$ | $E=\{-\mu B, +\mu B\}$ |

Take the magnetic aspect of the system — $Z_1$ refers to a single atom

$$Z_1=\sum_r e^{-E_r/kT}=e^{+\mu B/kT}+e^{-\mu B/kT}$$

$\boxed{\frac{\mu B}{kT} \text{ is dimensionless } x}$

All the complications are thrown into the heat bath.

Probability of finding it aligned to $+\vec{B}$

$$Pr_+=\frac{e^{-E_+/kT}}{Z_1}=\frac{e^{\mu B/kT}}{e^{\mu B/kT}+e^{-\mu B/kT}}=\frac{e^{x}}{e^{x}+e^{-x}}$$

aligned to $-\vec{B}$

$$Pr_-=\frac{e^{-E_-/kT}}{Z_1}=\frac{e^{-\mu B/kT}}{e^{\mu B/kT}+e^{-\mu B/kT}}=\frac{e^{-x}}{e^{x}+e^{-x}}$$

$x = \frac{\mu B}{kT}$ $\quad \lim_{T\to\infty} x = 0$ $\quad \lim_{T\to\infty} P_+ = 1/2$ $\quad \lim_{T\to\infty} P_- = 1/2$

$\therefore$ as $T\to\infty$ the system is completly disoriented

$$\langle \vec{\mu} \rangle = \sum_r P_r \mu_r = P_+ \mu_+ + P_- \mu_- = \left(\frac{e^x}{e^x+e^{-x}}\right)(\mu) + \left(\frac{e^{-x}}{e^{+x}+e^{-x}}\right)(-\mu)$$

$$= \mu \left(\frac{e^x - e^{-x}}{e^x+e^{-x}}\right) = \mu \tanh(x)$$

Note: $\cosh(x) = \frac{e^x+e^{-x}}{2}$ $\quad \sinh = \frac{e^x - e^{-x}}{2}$ $\quad \cos ix = \frac{e^{ix}+e^{-ix}}{2}$ memorize!

What is the total magnetic moment of the solid?

$\mathcal{M}_{net} = N\langle\mu\rangle = N\langle\mu\rangle \tanh(x)$ $\quad x = \mu B/kT$ $\quad$ z-component of moment

View as a system with magnetic moment of a single atom + the rest of the solid is a heat bath.

$$\langle E_1 \rangle = \langle -\vec{\mu}\cdot\vec{B}\rangle = -\langle\mu\rangle B$$

$$\langle E\rangle_{sys} = N\langle E_1\rangle = -N\langle\mu\rangle B = -NB\tanh(x)$$

$$Z_1 = e^x + e^{-x} = e^{\beta\mu B} + e^{-\beta\mu B}$$

$$\langle E_1 \rangle = \frac{-\partial Z_1}{\partial\beta} = -\frac{1}{Z}\frac{\partial Z_1}{\partial \beta} = \frac{\mu B e^{\beta\mu B} - \mu B e^{-\beta\mu B}}{e^{\beta\mu B} + e^{-\beta\mu B}} = \mu B \tanh(x)$$

~~at~~ for an intermediate temperature $\mathcal{M} = N\mu \tanh\left(\frac{\mu B}{kT}\right)$ $\quad \mu \sim \frac{e\hbar}{2mc}$ Bohr magnetron

Consider H atom

$B$ = current · area $\quad$ Amp · $m^2$ is the unit of $\mu$

$\hbar$ = J · m $\quad$ momentum × distance $\quad$ action

$\hbar = 1.055 \cdot 10^{-34}$ J · m
$e = 1.602 \cdot 10^{-19}$ C
$m = 9.11 \cdot 10^{-31}$ kg
$c = 3 \cdot 10^8$ m/s

$$\mu = \frac{e\hbar}{2mc} = \frac{(1.602\cdot10^{-19}\,C)(1.055\cdot10^{-34}\,J\cdot m)}{2(9.11\cdot10^{-31}\,kg)(3\cdot10^8\,m/s)} \sim 10^{-31}\,\frac{A}{m^2}$$

$kT = \frac{1}{40}$ eV at Room Temperature
$1.6\cdot10^{-19}$ J = 1eV

$kT = .4\cdot10^{-20}$ J at room temperature

$\frac{\mu B}{kT} = \frac{10^{-31} B}{10^{-21}} = 10^{-10} B \rightarrow 10^{-10} = \frac{\mu B}{kT}$ $\quad$ in a strong field $B \sim 1T = 10^5$ Gauss

for T=1K divide by 300, even at low temperatures $\frac{\mu B}{kT}$ is very small

Lecture 5 continued...

Ways to simplify the expression:

$\mathcal{M} = N\vec{\mu} = N\mu \tanh(x) \Rightarrow \quad x = \frac{\mu B}{kT}$

$x \ll 1$

$$\tanh(x) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \rightarrow \frac{(1+x) - (1-x)}{(1+x) + (1-x)} = \frac{2x}{2} = x \qquad x \ll 1$$

$$\therefore \quad \mathcal{M} \sim N\mu x = \left(\frac{N\mu B}{k}\right)\frac{1}{T}$$

Curie's Law: The net magnetic moment in the direction of $\vec{B}$ is inversely proportional to $T$

$$\boxed{\mathcal{M} \propto \frac{1}{T}} \qquad \text{when } \frac{\mu B}{kT} \ll 1$$

Can use this result to calibrate thermometers. It is a clever way to measure very low temperature fraction of K here.

It was found experimentally first.

There are at least two ways to find the magnetic moment in direction of B with statistical mechanics (with the heat bath system)

1. Specify equilibrium - When is the Helmholtz free energy a minimum (for a fixed temperature)?
2. When is entropy a maximum for a fixed temperature?

Start with maximum entropy + derive a condition that Helmholtz free energy is a minimum:

Thermodynamic Potentials

| | | | |
|---|---|---|---|
| T, V fixed | min(F) = equilibrium | $F = \langle E \rangle - TS = E - TS$ | Helmholtz |
| T, P fixed | min(G) = equilibrium | $G = E + PV - TS$ | Gibbs |
| | | $H = E + PV$ | Enthalpy |

If $F$ is a minimum at equilibrium we can get the same result.
Consider only the magnetic aspect of the solid:

$\rightarrow \ \rightarrow \ \rightarrow \ \rightarrow$ either parallel or antiparallel

$\rightarrow \ \leftarrow \ \leftarrow \ \rightarrow$

$n$ = # of atoms magnetic moment is parallel
$N$ = total # of atoms
$(N-n)$ = antiparallel

Energy $E = n(-\mu B) + (N-n)\mu B = (N-2n)\mu B$

Entropy $\quad S = k \ln \Omega \quad$ with $\Omega = \dfrac{N!}{n!(N-n)!}$ = # of microstates

How can we get $M_{net}$?
So many atoms we can use Stirlings approximation: $S(n) = k[N\ln N - n\ln n - (N-n)\ln(N-n)]$
for a given $n$, $S$ is a maximum.

$$S_{total} = \underset{\text{magnetic}}{S(n)} + \underset{\text{remaining vibrational motion of the atoms}}{S(R)}$$

$E(n) = (N-2n)B\mu$ at a fixed energy can use Boltzmann relation $\underbrace{E_{total} = E(n) + E_R = E_0}_{\text{fixed constant}}$

$$1 + \frac{\partial E_R}{\partial E_n} = 0$$

$$\frac{\partial S_{total}}{\partial n} = \frac{\partial S(n)}{\partial n} + \frac{\partial S_R}{\partial n} = 0$$

$$\frac{\partial S_n}{\partial n} = \frac{\partial S_n}{\partial E}\frac{\partial E}{\partial n} \qquad \frac{\partial S_R}{\partial n} = \frac{\partial S_R}{\partial E_R}\overset{-1}{\left(\frac{\partial E_R}{\partial E_n}\right)}\frac{\partial E_n}{\partial n} = -\frac{\partial S_R}{\partial E_R}\frac{\partial E_n}{\partial n}$$

$$\therefore \frac{\partial S_T}{\partial n} = \frac{\partial S_n}{\partial E}\cancel{\frac{\partial E}{\partial n}} - \frac{\partial S_R}{\partial E_R}\cancel{\frac{\partial E}{\partial n}} = 0$$

$$\frac{\partial S_n}{\partial E} = \frac{\partial S_R}{\partial E_R} = \frac{1}{T}$$ condition of equilibrium gives the same result

$$\frac{\partial S_n}{\partial E} = \frac{\partial S_n}{\partial n}\frac{\partial n}{\partial E} \rightarrow \frac{1}{T} = \frac{-1}{2\mu B}\frac{\partial S(n)}{\partial n}$$

$$\frac{\partial S_n}{\partial n} = k\left[-n\left(\frac{1}{n}\right) - \ln(n)1 - (N-n)\left(\frac{1}{N-n}\right)(-1) - \ln(N-n)(-1)\right] = k\ln\left\{\frac{N-n}{n}\right\}$$

$$\therefore \frac{1}{T} = -\left(\frac{1}{2\mu B}\right)k\ln\left(\frac{N-n}{n}\right) \rightarrow \ln\left(\frac{N-n}{n}\right) = -\frac{2\mu B}{kT} \rightarrow \frac{N}{n} = 1 + e^{-2\mu B/kT}$$

$$\rightarrow P_+ = \frac{n}{N} = \frac{e^{\mu B/kT}}{e^{\mu B/kT} + e^{-\mu B/kT}}$$

Same thing!

it is much easier to do with partition function.
Can easily generalize to $J = 1$ or $n$