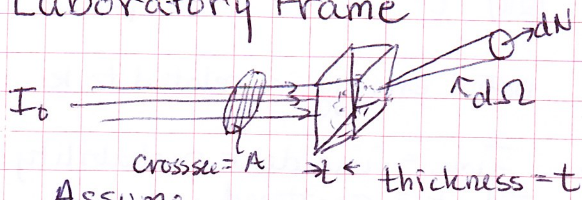
Scattering

Non-Relativistic Elastic Scattering
- alpha by Nuclues (Rutherford) = Coulomb
- nutron by proton = Nuclear

Laboratory Frame



$I_0 = \#\text{particles}/\text{area time}$

Assume
① incident particles are non-interacting
② single incident particle interacts with single target

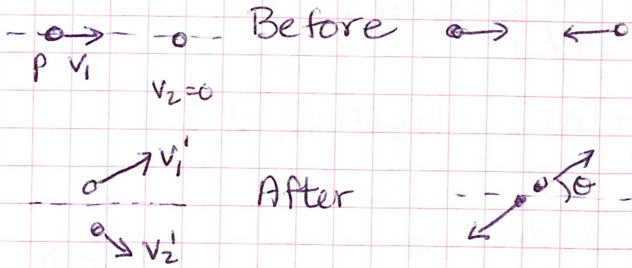
dN = # projectile particles observed per second

$dN \propto I_0 n A t$

$dN = d\sigma\, I_0 (nAt)$

$d\sigma = \frac{dN}{I_0 n A t}$ = "pure number" $= \frac{I(\theta,\phi)\,d\Omega}{I_0 n A t} \Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{I(\theta,\phi)}{I_0 n A t}}$

Lab Frame | CM Frame



$\boxed{d\sigma_L = d\sigma_{cm}}$

$\Rightarrow \frac{d\sigma_L}{d\Omega_L} = \frac{d\sigma_{cm}}{d\Omega_{cm}}\frac{d\Omega_{cm}}{d\Omega_L} = \frac{d\sigma_{cm}}{d\Omega_{cm}}\frac{\sin\theta_{cm}d\theta_{cm}d\phi_{cm}}{\sin\theta_L d\theta_L d\phi_L}$

$\beta = m_1/m_2 \qquad \phi_{cm} = \phi_L$

$$\boxed{\frac{d\sigma_L}{d\Omega_L} = \frac{d\sigma_{cm}}{d\Omega_{cm}}\frac{(1+\beta^2+2\beta\cos\theta_{cm})^{3/2}}{(1+\beta\cos\theta_{cm})}}$$

$m_1 v_1 = m_1 v_1' + m_2 v_2'$

$\frac{1}{2}m_1 v_1^2 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2$

$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$

$\vec{V}_{cm} = \dot{\vec{R}} = \frac{m_1\dot{\vec{r}}_1}{m_1+m_2} = \frac{m_1\vec{v}_1}{m_1+m_2}$

$\vec{v}_{1cm} = \vec{v}_1 - \vec{V}_{cm} = \frac{m_2\vec{v}_1}{m_1+m_2}$

$\vec{v}_{2cm} = 0 - \vec{V}_{cm} = -\frac{m_1\vec{v}_1}{m_1+m_2}$

$\vec{p}_{1cm} = m_1\vec{v}_{1cm} \quad \vec{p}_{2cm} = m_2\vec{v}_{2cm}$

$\vec{p}_{cm} = 0 = \vec{p}_{1cm} + \vec{p}_{2cm}$

$p_{1cm} = -p_{2cm}$

$p'_{1cm} = -p'_{2cm}$

$\frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2} = \frac{p_{1cm}^2}{2m_1} + \frac{p_2^2}{2m_2}$

$v_1'\cos\theta_L = v_{1cm}'\cos\theta_{1cm} + V_{cm}$

$v_1'\sin\theta_L = v_{1cm}'\sin\theta_{1cm}$

$\tan\theta_{1L} = \frac{v_{1cm}'\sin\theta_{1cm}}{v_{1cm}'\cos\theta_{1cm} + V_{cm}}$

$\tan\theta_L = \frac{\sin\theta_{1cm}}{\cos\theta_{1cm} + (V_{cm}/v_{1cm})}$

$\tan\theta_{1L} = \frac{\sin\theta_{1cm}}{\cos\theta_{1cm} + m_1/m_2}$

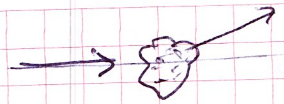
$\tan\theta_{1L} = \frac{2\sin(\theta_{1cm}/2)\cos(\theta_{1cm}/2)}{2\cos^2(\theta_{1cm}/2)}$

$\tan\theta_{1L} = \tan(\theta_{1cm}/2)$

$\boxed{\theta_{1L} = \theta_{1cm}/2}$

finite radius

Scattering - Plane Wave incident prove $\left(\frac{d\sigma}{d\Omega}\right)_{cm} = |f_k(\theta,\phi)|^2$



Before: $\psi_{inc} = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}}$ initial packet is approx by plane wave

After: $\psi_{scat} = \lim_{r\to\infty} \psi_k^{(+)}(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\left\{e^{i\vec{k}\cdot\vec{r}} + f_k(\theta,\phi)\frac{e^{i\vec{k}\cdot\vec{r}}}{r}\right\}$

Going to prove $\left(\frac{d\sigma}{d\Omega}\right)_{cm} = |f_k(\theta,\phi)|^2$ note: energy is related to k

$d\sigma = \frac{(\vec{j}_{scatt}\cdot\hat{r})r^2 d\Omega}{j_{inc}}$ ~~$j_{scatt}\cdot\hat{r}$~~ j_{inc} $\vec{j}_{inc}$ = incident probability density, $\vec{j}_{scat}$ = scattered

$\psi_i = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}}$ $\psi_s = \frac{1}{(2\pi)^{3/2}} f_k(\theta,\phi)\frac{e^{i\vec{k}\cdot\vec{r}}}{r}$ $\nabla\psi_s = \left(\frac{1}{2\pi}\right)^{3/2} f_k(\theta,\phi)\frac{ike^{i\vec{k}\cdot\vec{r}}}{r} + \cancel{\mathcal{O}\left(\frac{1}{r^2}\right)}^{\to 0}$

$\vec{j} = \frac{\hbar}{2\mu i}\left\{\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right\}$ $\vec{\nabla}\cdot\vec{j} = -\frac{\partial\rho}{\partial t}$ $\hbar k$ = velocity

$\vec{j}_{inc} = \frac{\hbar}{2\mu i}\left\{\left(\frac{1}{2\pi}\right)^{3/2} e^{-i\vec{k}\cdot\vec{r}}\, i\vec{k}\left(\frac{1}{2\pi}\right)^{3/2} e^{i\vec{k}\cdot\vec{r}} - \left(\frac{1}{2\pi}\right)^{3/2} e^{i\vec{k}\cdot\vec{r}}(-i\vec{k})\left(\frac{1}{2\pi}\right)^{3/2} e^{-i\vec{k}\cdot\vec{r}}\right\} = \frac{\hbar}{2\mu} 2\vec{k}\left(\frac{1}{2\pi}\right)^3 = \frac{\hbar\vec{k}}{2^3\pi^3\mu}$

$\vec{j}_{scatt} = \frac{\hbar}{2\mu i}\left\{\left(\frac{1}{2\pi}\right)^{3/2} f_k^*(\theta,\phi)\frac{e^{-ikr}}{r}\left(\frac{1}{2\pi}\right)^{3/2} f_k(\theta,\phi)\, ik\frac{e^{ikr}}{r} - \left(\frac{1}{2\pi}\right)^{3/2} f_k(\theta,\phi)\frac{e^{ikr}}{r}\left(\frac{1}{2\pi}\right)^{3/2} f_k^*(\theta,\phi)(-ik)\frac{e^{-ikr}}{r}\right\}\hat{r}$

$\vec{j}_{scat} = \frac{\hbar}{2\mu i} 2\left\{|f_k(\theta,\phi)|^2\left(\frac{1}{2\pi}\right)^3\frac{1}{r^2}\, ik\right\}\hat{r} = \frac{\hbar}{\mu}\left(\frac{1}{2\pi}\right)^3\frac{k}{r^2}|f_k(\theta,\phi)|^2\hat{r}$

$d\sigma = \frac{(j_s\cdot\hat{r})r^2 d\Omega}{j_{inc}} = \frac{(\hbar k/\mu)|f_k(\theta,\phi)|^2(1/2\pi)^3 d\Omega}{(\hbar k/\mu)(1/2\pi)^3} = |f_k(\theta,\phi)|^2 d\Omega$

$\therefore \boxed{\frac{d\sigma}{d\Omega} = |f_k(\theta,\phi)|^2}$ Scattering amplitude = f_k Q.E.D.

Scattering

To show there are solutions of the type, solve TISE via Green's Theorem

$$(\nabla^2 + k^2)\Psi(\vec{r}) = U\Psi(\vec{r}) \qquad U = \frac{2\mu}{\hbar^2}V(\vec{r}) \qquad k^2 = 2\mu E/\hbar^2$$

Every partial differential equation has a Green's function
If you have the Green's function, you have the solution to the TISE.

Definition of Green's Function: $(\nabla^2 + k^2)G(\vec{r},\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,')$

Assume $G(\vec{r},\vec{r}\,') = G(\vec{r}-\vec{r}\,')$ replace $\vec{r}-\vec{r}\,'$ by $\vec{r}$

$$(\nabla^2 + k^2)G(\vec{r}) = -4\pi\delta(\vec{r})$$

Any reasonable function can be written as a fourier integral

$$G(\vec{r}) = \int g(\vec{k}')e^{-i\vec{k}'\cdot\vec{r}}\,d^3k' \qquad \delta(\vec{r}) = \left(\frac{1}{2\pi}\right)^3\int e^{i\vec{k}'\cdot\vec{r}}\,d^3k' \qquad k \neq k'$$

Can take the Laplacian inside:

$$\int g(\vec{k}')[\nabla^2 + k^2]e^{i\vec{k}'\cdot\vec{r}}\,d^3k' = -4\pi\left(\frac{1}{2\pi}\right)^3\int e^{i\vec{k}'\cdot\vec{r}}\,d^3k'$$

$$g(\vec{k}')(-k'^2 + k^2) = \frac{-4\pi}{(2\pi)^3} \qquad g(\vec{k}') = \frac{-1}{2\pi^2}\frac{1}{k^2-k'^2} = \left(\frac{1}{2\pi^2}\right)\frac{1}{(k'^2-k^2)}$$

$$(\nabla^2 + k^2)\int G(\vec{r}-\vec{r}\,')U(\vec{r}\,')\Psi(\vec{r}\,')\,d^3r' = -4\pi\int \delta(\vec{r}-\vec{r}\,')U(\vec{r}\,')\Psi(\vec{r}\,')\,d^3r'$$

$$= -4\pi\left(\frac{-1}{4\pi}\right)U(\vec{r})\Psi(\vec{r}) = U(\vec{r})\Psi(\vec{r})$$

$$\underbrace{\Psi(\vec{r}) = \underbrace{\left(\frac{1}{2\pi}\right)^{3/2}e^{i\vec{k}\cdot\vec{r}}}_{\text{particular solution = homo solution}} - \frac{1}{4\pi}\int G(\vec{r}-\vec{r}\,')U(\vec{r}\,')\Psi(\vec{r}\,')\,d^3r'}_{\text{formal solution of the equation}}$$

Scattering — First Born Approximation

Exact Solution: $\left(\frac{1}{2\pi}\right)^{3/2} f_k(\theta,\phi) = \frac{-\mu}{2\pi\hbar^2}\int e^{-i\vec{k}'\cdot\vec{r}'} V(\vec{r}')\,\psi_k^+(\vec{r}')\,d^3r'$

First Born Approx: Replace $\psi_k^+(\vec{r}) \to \left(\frac{1}{2\pi}\right)^{3/2} e^{i\vec{k}\cdot\vec{r}}$

Solution of the homogeneous eq: $\cancel{\left(\frac{1}{2\pi}\right)^{3/2}} f_k(\theta,\phi) = -\frac{\mu}{2\pi\hbar^2}\cancel{\left(\frac{1}{2\pi}\right)^{3/2}}\int V(\vec{r}')\,e^{i(\vec{k}-\vec{k}')\cdot\vec{r}'}d^3r'$

let $\vec{q} = \vec{k}' - \vec{k} \Rightarrow$ $\quad f_k(\theta,\phi) \propto \int V(\vec{r}')\,e^{i\vec{q}\cdot\vec{r}'}d^3r'$ (a fourier tranform)

related to the momentum transfered with a unit of $\hbar$

Note: Once you know the scattering you know something about the potential.

1st Born OK when
1) you can neglect the scattering amplitude
2) KE is high
3) Potential is weak

Example: Once you have the scattering, you have the answer

$$\frac{d\sigma}{d\Omega} = |f_k(\theta,\phi)|^2$$

Some potentials:

$V(r) = \frac{g e^{-\alpha r}}{r}$ Yukawa π-meson nucleuslike pt charge 10^{-18}cm

$V(r) = \frac{g}{r}$ Coulomb is a special case when $\alpha \to 0$

try Yukawa

$$f_k(\theta,\phi) = -\frac{\mu}{2\pi\hbar^2} g\int_{V'} \frac{e^{-\alpha r'}}{r'} e^{-i\vec{q}\cdot\vec{r}'} r'^2 dr'\sin\theta' d\theta' d\phi'$$

$$= -\frac{\mu}{\cancel{2\pi}\hbar^2} g\, 2\pi \int_0^{\pi}\int_0^{r_0} e^{-\alpha r'\cos\theta}\; r'dr'\sin\theta d\theta'$$

What's the condition for the validity of the 1st Born Approximation?

$$\psi_s = -\frac{1}{4\pi}\frac{2\mu}{\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} V(\vec{r}')\left(\frac{1}{2\pi}\right)^{3/2} e^{i\vec{k}\cdot\vec{r}'}d^3r' \qquad \psi_i = \left(\frac{1}{2\pi}\right)^{3/2} e^{i\vec{k}\cdot\vec{r}}$$

$\psi_s \ll \psi_i \qquad \psi_s$ is max as $r \to 0$

$\therefore$ if $\psi_s(\vec{r}\to 0) \ll \psi_i$ then 1st Born Approx is valid

$$\boxed{\left|\frac{\psi_s(r=0)}{\psi_i(r=0)}\right| \ll 1}$$ Condition for validity ~~$kr \ll 0$~~

$$\frac{\psi_s}{\psi_i} = \frac{r}{2\pi\hbar^2}\int \frac{e^{ikr'}V(\vec{r}')}{r'} e^{ikr'\cos\theta} r'^2 dr'\sin\theta' d\theta' d\phi' = \left|\frac{\mu}{\hbar^2}\int \frac{e^{ikr'}}{r'} V(r')r'^2dr'\,\frac{(e^{ikr'}-e^{-ikr'})}{ikr'}\right| \ll 1$$

$$\frac{2\mu}{\hbar k}\left|\int e^{ikr'}V(r')\sin(kr')dr\right| \ll 1 \Rightarrow kr' \ll 0$$

Low Energy & High Energy Condition for Validity

Low Energy Limit

$$E = \frac{\hbar^2 k^2}{2\mu}$$

$kr' \ll 1 \qquad kr_0 \ll 1$

$$\frac{2\mu}{\hbar^2 k}\left|\int e^{ikr'} V(r') \sin(kr')\, dr'\right| \ll 1$$

$$\boxed{\lim_{kr' \to 0} \frac{2\mu}{\hbar^2 k}\int V(r')\, kr'\, dr' \ll 1}$$

Example: Square Well Potential

$$\rightarrow \frac{2\mu}{\hbar^2 k} V_0 k \int_0^{r_0} r'\, dr' \ll 1$$

$$\frac{\cancel{2}\mu}{\hbar^2} \cdot \frac{r_0^2}{\cancel{2}} V_0 \ll 1$$

$$\boxed{V_0 \ll \frac{\hbar^2}{\mu r_0^2}} \qquad \boxed{\frac{\mu V_0 r_0^2}{\hbar^2} \ll 1}$$

If a particle is confined in a small potential of width $\emptyset$ the momentum must be of at least $\hbar/r_0$

$KE \sim \langle p_x^2 \rangle$

and $PE \ll KE$

High Energy Limit

$kr_0 \gg 1$

$$\frac{2\mu}{\hbar^2 k}\left|\int_0^\infty V(r') e^{ikr'} \sin(kr')\, dr'\right| \ll 1 \leftarrow$$

get this condition by $\left|\frac{\psi_s(r=0)}{\psi_i(r=0)}\right| \ll 1$

$$\lim_{kr_0 \to \infty} e^{ikr'} \sin(kr') = e^{ikr'}\left(\frac{e^{ikr'} - e^{-ikr'}}{2i}\right) = \frac{1}{2i}\left(e^{2ikr'} - 1\right)$$

oscillates rapidly about $\emptyset$ ∴ integral goes to $\emptyset$

$$\frac{2\mu}{\hbar^2 k}\left|\int_0 \frac{V(r')}{2i}\left(\cancel{e^{2ikr'}}^{\,0} - 1\right) dr'\right| = \frac{2\mu}{\hbar^2 k}\left|\int V(r')\left(\frac{-1}{2i}\right) dr'\right|$$

$$= \boxed{\frac{\mu}{\hbar^2 k}\left|\int V(r')\, dr'\right| \ll 1}$$

Example: Square Well Potential

$$\frac{\mu}{\hbar^2 k} V_0 \int_0^{r_0} dr' = \boxed{\frac{\mu_0 V_0 r_0}{\hbar^2 k} \ll 1} \quad \text{or} \quad \frac{\mu V_0}{\hbar^2} \ll \frac{k}{r_0} \qquad \boxed{\frac{\mu_0 V_0 r_0^2}{\hbar^2} \ll kr_0}$$

Low Energy $\qquad \frac{\mu V_0 r_0^2}{\hbar^2} \ll 1$

High Energy $\qquad \frac{\mu V_0 r_0^2}{\hbar^2} \ll kr_0$

much easier to satisfy at high energies than at low energies

Scattering   Calculate Phase Shift

How to Calculate Phase Shift

1. Solve SE with Potential
2. Potential causes phase shift

$$f_k(\theta) = \sum_{l=0}^{\infty} (2l+1)\left(\frac{1}{2ik}\right)\left(e^{2i\delta_l} - 1\right) P_l(\cos\theta)$$

$$\frac{1}{2i}\left(e^{2i\delta_l} - 1\right) = \frac{e^{i\delta_l}}{2i}\left(e^{i\delta_l} - e^{-i\delta_l}\right) = e^{i\delta_l}\sin\delta_l$$

only for central potentials

$$f_k(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l\, P_l(\cos\theta)$$

if no potential $\delta_l \to 0$ $f_k(\theta) \to 0$ no scattered part

δ_l is only there because of the potential - it is an infinite sum
at low energies only the $l=0$ part contribute $P_l(\cos\theta) = 1$

if $f_k(\theta)$ is independent of θ → completely isotropic

$$\frac{d\sigma}{d\Omega} = |f_k(\theta)|^2 = \frac{1}{k^2}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}(2l+1)(2l'+1)e^{i(\delta_l - \delta_{l'})}\sin\delta_l \sin\delta_{l'} P_l(\cos\theta)P_{l'}(\cos\theta)$$

$$\sigma_T = \int\left(\frac{d\sigma}{d\Omega}\right)d\Omega = \frac{1}{k^2}\underbrace{2\pi}_{\int d\phi}\sum_l\sum_{l'}(2l+1)(2l'+1)e^{i(\delta_l - \delta_{l'})}\sin\delta_l\sin\delta_{l'}\int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta$$

$d\Omega = \sin\theta\, d\theta\, d\phi$

$$\int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\,d(\cos\theta) = \left(\frac{2}{2l+1}\right)\delta_{ll'}$$

$$\sigma_T = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l = \sum_{l=0}^{\infty}\sigma_l \Rightarrow \sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l$$

Scattering — Optical Theorem

Conservation of Probability

$$\sigma_T = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell \qquad f_k(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell\, e^{i\delta_\ell} P_\ell(\cos\theta)$$

What is $f_k(\theta=0)$? forward scattering amplitude

$$f_k(0) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell$$

$$\sigma_T = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell = \frac{4\pi}{k}\,\mathrm{Im}(f_k(0))$$ derive from cons. of probability

Conservation of Particles – only works in limited circumstances

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \qquad \rho = \psi^*\psi \qquad \vec{j} = \frac{\hbar}{2\mu}\{\psi^*\vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi\}$$

in an energy eigenstate ρ is independent of time calculate $\psi^*\psi$ is time independent

$\nabla j \to 0$

$$\frac{\partial}{\partial t}\int \rho\, d^3x + \int \vec{\nabla}\cdot\vec{j}\, d^3x = 0$$

can write as $\int \frac{\partial\rho}{\partial t} d^3x + \oint j_{normal}\, dA = 0$ for stationary states this is zero only works for stationary state

$$\vec{j} = \frac{\hbar}{2\mu i}\{\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\} = \frac{\hbar}{\mu}\mathrm{Im}(\psi^*\vec{\nabla}\psi)$$ imaginary part of complex # close find cenered @ origin then the normal component

can write $\psi = \phi + \chi$

soln for free part incident — scattered part

0 stationary state of free particle

$$\mathrm{Im}\oint(\phi+\chi)^*\frac{\partial}{\partial r}(\phi+\chi)\, r^2 d\Omega = 0 = \mathrm{Im}\oint\phi^*\left(\frac{\partial\phi}{\partial r}\right)r^2 d\Omega + \mathrm{Im}\oint\chi^*\left(\frac{\partial\chi}{\partial r}\right)r^2 d\Omega$$
$$+ \mathrm{Im}\oint\phi^*\left(\frac{\partial\chi}{\partial r}\right)r^2 d\Omega + \mathrm{Im}\oint\chi^*\left(\frac{\partial\phi}{\partial r}\right)r^2 d\Omega$$

$r \to \infty$

$$\lim_{r\to\infty}(\phi+\chi) = \left(\frac{1}{2\pi}\right)^{3/2}\Big\{\underbrace{e^{i\vec{k}\cdot\vec{r}}}_{\phi} + \underbrace{f_k(\theta,\phi)\frac{e^{ikr}}{r}}_{\chi}\Big\}$$

$$\frac{\partial\chi}{\partial r} = f_k(\theta,\phi)\left(\frac{-1}{r^2}e^{ikr} + ik\frac{e^{ikr}}{r}\right) = f_k(\theta,\phi)\frac{ik}{r}e^{ikr} = ik\chi$$

$$\chi^*\frac{\partial\chi}{\partial r}r^2 d\Omega = \mathrm{Im}\oint\left(\frac{1}{2\pi}\right)^{3/2} f_k^*(\theta,\phi)\frac{e^{-ikr}}{r}\left(\frac{1}{2\pi}\right)^{3/2} f_k(\theta,\phi)\frac{ik}{r}e^{ikr}\, r^2 d\Omega$$

$$= \mathrm{Im}\, ik\oint\left(\frac{1}{2\pi}\right)^3 f_k^2(\theta,\phi)\, d\Omega = \left(\frac{1}{2\pi}\right)^3\sigma_T$$

Scattering - Partial Wave Expansion

Partial Wave Expansion

Orbital angular momentum is conserved during scattering

Classically $\vec{L}_z = \vec{r} \times \vec{p} = 0 \quad m\vec{v}_z \times \vec{z}$

$$\Psi_{inc} = \left(\frac{1}{2\pi}\right)^{3/2} e^{ikz} = \left(\frac{1}{2\pi}\right)^{3/2} \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\theta)$$

← Spherical Bessel function

$j_l = Y_l^0(\theta, \phi)$ $m=0$ $L_z|\Psi\rangle = 0$

Asymptotic Form

$$\lim_{r\to\infty} j_l(kr) = \frac{1}{kr}\sin(kr - l\pi/2) \qquad \lim_{r\to\infty} \eta_l(kr) = \frac{-1}{kr}\cos(kr - l\pi/2)$$

$\lim_{\substack{r\to 0 \\ m=0}} \eta_l(kr) \to \infty$ doesn't work for spherical neuman

$$\lim_{\rho\to 0} j_l(\rho) = \frac{\rho^l}{(2l+1)!!} \qquad \lim_{\rho\to 0} \eta_l(\rho) = -\frac{(2l+1)!!}{\rho^{l+1}}$$

$$\lim_{r\to\infty} \Psi_k^+(\vec{r}) = \left(\frac{1}{2\pi}\right)^{3/2} \left\{ e^{ikz} + f_k(\theta)\frac{e^{ikr}}{r} \right\} \qquad f_k(\theta) = \sum_{l=0}^{\infty}(2l+1)a_l(k)P_l(\cos\theta)$$

if it is a function of θ, it can be expanded in terms of Legendre Polynomials (complete)

Suppose solving S.E. with potential energy $\to 0$ (very far)

$$R_{nl}(r)\, Y_l^m(\theta,\varphi) \underset{m=0}{\longrightarrow} R_{ln}(r) P_l(\cos\theta) \qquad \text{T.I.S.E.}$$

$$\lim_{r\to\infty}\left\{A_l j_l(kr) + B_l \eta_l(kr)\right\} = A_l\left\{j_l(kr) + \frac{B_l}{A_l}\eta_l(k)\right\} = A_l\left\{j_l(k) + \tan\delta_l\, \eta_l(kr)\right\}$$

$$= A_l\left\{\frac{1}{kr}\sin(kr - l\pi/2) + \tan\delta_l \frac{\cos(kr - l\pi/2)}{kr}\right\}$$

$$= \frac{A_l}{kr\cos\delta_l}\left\{\cos\delta_l \sin(kr - l\pi/2) + \sin\delta_l\cos(kr - l\pi/2)\right\}$$

$$= \frac{A_l}{kr\cos\delta_l}\sin(kr - l\pi/2 + \delta_l) \quad \text{with } \tan\delta_l = \frac{B_l}{A_l}$$

if there is no potential, j_l is allowed then $\sin(kr - l\pi/2)$
the potential causes a phase shift

$-\phi$ repulsive
$+\phi$ attractive

let $\dfrac{A_\ell}{\cos\delta_\ell} = C_\ell$ $\quad \therefore \quad R_{n\ell} = \dfrac{C_\ell}{kr}\sin(kr - \ell\pi/2 + \delta_\ell)$

$$\sum_{\ell=0}^{\infty} C_\ell \left(\frac{1}{kr}\right)\sin(kr - \ell\pi/2 + \delta_\ell)P_\ell(\cos\theta) = \left(\frac{1}{2\pi}\right)^{3/2}\left\{\sum_{\ell=0}^{\infty} i^\ell(2\ell+1)j_\ell(kr)P_\ell(\cos\theta)\right\}$$

$$C_\ell\left(\frac{1}{kr}\right)\left[\frac{e^{i(kr-\ell\pi/2+\delta_\ell)} - e^{-i(kr-\ell\pi/2+\delta_\ell)}}{2i}\right]P_\ell(\cos\theta) = \left(\frac{1}{2\pi}\right)^{3/2}\left\{\sum(i)^\ell(2\ell+1)\frac{\sin(kr-\ell\pi/2)}{kr}P_\ell(\cos\theta)\right\}$$

$$+ \sum(2\ell+1)a_\ell(k)P_\ell(\cos\theta)\frac{e^{ikr}}{r}$$

$$= \left(\frac{1}{2\pi}\right)^{3/2}\left\{\sum i^\ell(2\ell+1)\left(\frac{e^{i(kr-\ell\pi/2)} - e^{-i(kr-\ell\pi/2)}}{2ikr}\right)P_\ell(\cos\theta)\right.$$

$$\left. + \sum(2\ell+1)a_\ell(k)P_\ell(\cos\theta)\frac{e^{ikr}}{r}\right\}$$

equate coef of $\dfrac{e^{-ikr}}{kr}P_\ell(\cos\theta)$ on both sides

$$= \sum(2\ell+1)\frac{e^{ikr} - e^{i\ell\pi}e^{-ikr}}{2ikr}P_\ell(\cos\theta)$$

also using $i = e^{i\pi/2} \rightarrow \cos\pi/2 + i\sin\pi/2$ $\quad i^\ell = e^{i\ell\pi/2}$

$$\Rightarrow -\frac{C_\ell}{2i}e^{i\ell\pi/2}e^{-i\delta_\ell} = -\left(\frac{1}{2\pi}\right)^{3/2}\frac{e^{i\ell\pi}}{2i}(2\ell+1)$$

$$C_\ell = \left(\frac{1}{2\pi}\right)^{3/2}e^{i\ell\pi/2}e^{i\delta_\ell}(2\ell+1)$$ expansion coefficients

$$C_\ell\frac{e^{i\pi\ell/2}}{2i}e^{i\delta_\ell} = \left\{\frac{(2\ell+1)}{2i} + (2\ell+1)ka_\ell(k)\right\}\left(\frac{1}{2\pi}\right)^{3/2}$$

Substitute C_ℓ

$$\frac{e^{2i\delta_\ell}}{2i} = \frac{1}{2i} + ka_\ell(k)$$

$$ka_\ell(k) = \frac{1}{2i}\left(e^{2i\delta_\ell} - 1\right)$$

$$a_\ell(k) = \frac{1}{2ik}\left(e^{2i\delta_\ell} - 1\right) \qquad 19.5.6$$

Scattering - Optical Theorem

$$\Im m \oint \phi^* \frac{\partial \chi}{\partial r} r^2 d\Omega + \Im m \oint \chi^* \frac{\partial \phi}{\partial r} r^2 d\Omega$$

$$= \Im m \oint \left(\phi^* \frac{\partial \chi}{\partial r} + \chi^* \frac{\partial \phi}{\partial r} \right) r^2 d\Omega$$

$$= \Im m \oint \left(\phi^* \frac{\partial \chi}{\partial r} - \chi \frac{\partial \phi^*}{\partial r} \right) r^2 d\Omega$$

$$= \Im m \oint \{ \phi^* \nabla \chi - \chi \vec{\nabla} \phi^* \} \cdot \hat{n} dA$$

$$= \Im m \int_V \nabla \cdot (\phi^* \nabla \chi - \chi \nabla \phi^*) dV$$

Can write as

$$\Im m \oint \{ \phi^* \nabla \psi - \psi \nabla \phi^* \} \cdot \hat{n} dA = \Im m \int_V \nabla \cdot (\phi^* \vec{\nabla} \psi - \psi \vec{\nabla} \phi^*) dV$$

$$= \Im m \int (\phi^* \nabla^2 \psi - \psi \nabla^2 \phi) dV$$

Can substitute $(\nabla^2 + k^2)\phi = 0$

$$(\nabla^2 + k^2)\psi = U(r)\psi$$

$$U(r)\psi = \frac{2\mu}{\hbar^2} V(\vec{r})\psi(\vec{r})$$

$$= \int (\phi^* (U\psi - k^2\psi) + \psi(k^2)\phi^*) dV$$

$$= \int \phi^* U \psi dV$$

$$= \int \phi^* \frac{2\mu}{\hbar^2} V(\vec{r}) \psi(\vec{r}) dV$$

Scattering - Phase Shift for Hard Sphere

Calculation of the phase-shift δ_ℓ for a hardsphere

$$V(r) = \begin{cases} \infty & r<r_0 \\ 0 & r>r_0 \end{cases}$$

Classical $\sigma = \pi r_0^2$

QMech: to calculate phase shift solve S.E. $\psi(r_0)=0$

$r>r_0$ $\quad -\frac{\hbar^2}{2\mu}\nabla^2\psi = E\psi \qquad (\nabla^2 + k^2)\psi = 0$

Central Potential Solution: $\psi(\vec{r}) = R_{n\ell} Y_\ell^m \qquad R_{n\ell} = \frac{U_{n\ell}}{r}$

$$\frac{d^2u}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - V - \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2}\right]u(r) = 0$$

Can solve for $U(r)$ or $R(r)$

$R_{n\ell}(r) = A_\ell j_\ell(kr) + B_\ell \eta_\ell(kr)$ $\quad r>r_0$ (not $r=0$) define $\tan\delta_\ell = -\frac{B_\ell}{A_\ell}$

$R_{n\ell} = A_\ell\left[j_\ell(kr) - (\tan\delta_\ell)\eta_\ell(kr)\right]$

$R_{n\ell}(r_0) = 0$ since it is a hard sphere, the radial part must also vanish.

$j_\ell(kr_0) - \tan\delta_\ell\, \eta_\ell(kr_0) = 0 \Rightarrow \boxed{\tan\delta_\ell = \frac{j_\ell(kr_0)}{\eta_\ell(kr_0)}}$ in general you have to know the spherical Bessel function easy $\ell=0$

only small values of ℓ contribute

$$\tan\delta_0 = \frac{j_0(kr_0)}{\eta_0(kr_0)} = \frac{\frac{\sin(kr_0)}{kr_0}}{-\frac{\cos(kr_0)}{kr_0}} = -\tan(kr_0) = \tan(-kr_0) \Rightarrow \boxed{\delta_0 = -kr_0}$$

$\sigma_T = 4\pi k^2 \sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell$ $\qquad$ semiclassical arg $|\vec{L}| = |\vec{r}\times\vec{p}| = pr_0 \sim \hbar \ell_{max}$

the maximum ℓ for classical scattering $\ell_{max} \sim \frac{pr_0}{\hbar} \Rightarrow \frac{\hbar k r_0}{\hbar} = kr_0$

if $kr_0 \ll 1$ then $\ell > 1$ will not contribute to the sum

$$\sigma_T = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell \Rightarrow \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi k^2 r_0^2}{k^2} = 4\pi r_0^2$$

$\leftarrow$ QMech has 4x classical result

for higher values of ℓ, you need more to work out

$$\lim_{\rho\to 0} j_\ell(\rho) = \frac{\rho^\ell}{(2\ell+1)!!} \qquad \lim_{\rho\to 0}\eta_\ell(\rho) = -\frac{(2\ell-1)!!}{\rho^{\ell+1}} \qquad kr_0 \ll 1$$

$$\tan\delta_\ell \sim \frac{\frac{(kr_0)^\ell}{(2\ell+1)!!}}{-\frac{(2\ell-1)!!}{(kr_0)^{\ell+1}}} = -\frac{(2\ell-1)!!}{(2\ell+1)!!}(kr_0)^{2\ell+1} = \frac{-1}{(2\ell+1)}(kr_0)^{2\ell+1}$$

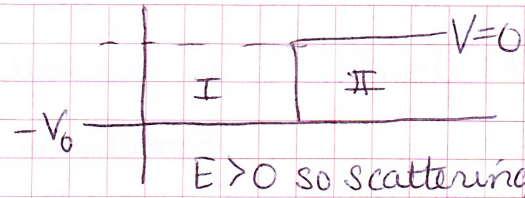
$$\delta_\ell \sim \frac{1}{(2\ell+1)}(kr_0)^{2\ell+1} \qquad \sigma_T = \sum \sigma_\ell = \sum_{\ell=0}^{\infty}\frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell = \sum \frac{4\pi}{k^2}(2\ell+1)\left(\frac{1}{2\ell+1}\right)(kr_0)^{4\ell+2} = \sum_{\ell=0}^{\infty}\frac{4\pi (kr_0)^{4\ell+1}}{k^2 (2\ell+1)}$$

Phase Shift – Spherical Square Well Potential Scattering

Spherical Square Well

$r<r_0 \quad V(r) = -V_0$

$r>r_0 \quad V(r) = 0$



$E>0$ so scattering

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi_I - V_0\psi_I = E\psi_I$$

only Bessel because $r=0$

$$(\nabla^2 + k_1^2)\psi_I = 0 \qquad k_1^2 = \frac{2\mu}{\hbar^2}(E+V_0) \qquad \psi_I = R_{nl}Y_l^m \Rightarrow A_l j_l = R_{nl}$$

$$R_{II}(r) = A_l'\left[j_l + \frac{B_l'}{A_l'}\,n_l(k_2 r)\right] = A_l'\left[j_l(k_2r) - \tan\delta_l\, n_l(k_2 r)\right]$$

for low energy just consider $l=0$ term

If you want you can also work with $u(r) = rR(r)$

$$\frac{d^2u}{dr^2} + \frac{2\mu}{\hbar^2}\left\{E - V - \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}\right\}u = 0$$ if you solve for $l=0$, it is simple

$$\frac{d^2u_I}{dr^2} + \frac{2\mu}{\hbar^2}(E+V_0)u_I = 0 \qquad k_1^2 = \frac{2\mu}{\hbar^2}(E+V_0)$$

$$\frac{d^2u_I}{dr^2} + k_1^2 u_I = 0 \Rightarrow u_I = A\sin(kr)$$ (no cos because $r=0,\ u_I=0$)

$$u_{II} = B\sin(kr) + C\cos(kr) = B\left[\sin(kr) + \frac{C}{B}\cos(kr)\right] = B\left[\sin kr - \tan\delta_l \cos kr\right]$$

$$= \frac{B}{\cos\delta_l}\left[\cos\delta_l \sin kr - \sin\delta_l \cos kr\right] = \frac{B}{\cos\delta_l}\sin(kr-\delta_l)$$

u, u' are continuous can use log derivatives $\frac{1}{u}\frac{\partial u}{\partial r}$

$$\frac{1}{u_I}\frac{du_I}{dr} = \frac{1}{A\sin(k_1r)}Ak_1\cos(k_1r) = \frac{k_1}{\tan(k_1r)} = k_1\cot(k_1r)$$

$$\frac{1}{u_{II}}\frac{du_{II}}{dr} = \frac{1}{C\sin(k_2r-\delta_0)}Ck_2\cos(k_2r-\delta_0) = k_2\cot(k_2r-\delta_0)$$

equate them $\quad k_1\cot(k_1r_0) = k_2\cot(k_2r_0-\delta_0)$

$$\boxed{\delta_0 = k_2r_0 - \cot^{-1}\left(\frac{k_1}{k_2}\cot(k_1r_0)\right) \qquad k_1>k_2}$$

definitions $k_1 = k_2$ if $V_0 = 0 \quad \cancel{V(r) = V_0\delta(r-r_0)}$

Scattering - Resonance Scattering

$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell$ is generally small at low energies

Sometimes can go up very rapidly

$\sin^2\delta_\ell$ is max at $\delta_\ell = (m+1)\pi/2$

small energy interval

$$\delta_\ell = \delta_{bkg} + \tan^{-1}\left(\frac{\Gamma/2}{E_0 - E}\right) \Rightarrow \delta_\ell = \tan^{-1}\left(\frac{\Gamma/2}{E_0 - E}\right)$$

δ_{bkg} can be neglected

$$\sec^2\delta_\ell = 1 + \tan^2\delta_\ell = 1 + \frac{(\Gamma/2)^2}{(E_0-E)^2} = \frac{(E_0-E)^2 + (\Gamma/2)^2}{(E_0-E)^2} = \sec^2\delta_\ell$$

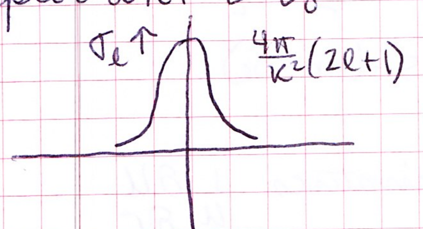
$$\cos^2\delta_\ell = \frac{(E_0-E)^2}{(E_0-E)^2 + (\Gamma/2)^2}$$

$$\sin^2\delta_\ell = 1 - \cos\delta_\ell = 1 - \frac{(E_0-E)^2}{(E_0-E)^2+(\Gamma/2)^2} = \frac{(\Gamma/2)^2}{(E_0-E)^2+(\Gamma/2)^2}$$

becomes 1 when $E = E_0$

so the partial cross-section dominates $\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\left\{\frac{(\Gamma/2)^2}{(E_0-E)^2+(\Gamma/2)^2}\right\}$

plot WRT $E - E_0$



even function

Breit - Wigner Formula

Lorentzian function