Today, we will solve some of the homework problems in class.

Hamiltonian:

$\hat{H} = \frac{p^2}{2m} + \frac{k^2 x^2}{2} = \hbar\omega_0\left(a^+ a + \frac{1}{2}\right)$ where $\omega_0^2 = \frac{k}{m}$ and $\beta^2 = \frac{m\omega_0}{\hbar}$

$x = \frac{a+a^+}{\sqrt{2}\beta}$ and $p = \frac{a-a^+}{\sqrt{2}\beta}\frac{m\omega_0}{i}$

$$\left.\boxed{\begin{array}{l} a^+\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle \\ a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle \end{array}}\right\}\text{Memorize This!}$$

Also, we learned the eigenfunction in Real space and to express it as:

$\varphi_n = A_n\left(\xi - \frac{\partial}{\partial\xi}\right)^n e^{-\xi^2/2}$

$\xi = \beta x$

Look at problem 7.8. We can prove the parity operator operates on eigenfunction:

$\hat{\wp}\varphi_n = (-1)^n \varphi_n$

$\hat{\wp}\varphi_n = \varphi_n(-x)$

$\hat{\wp}\varphi_n(\xi) = \varphi_n(-\xi) = A_n\left(-\xi - \frac{\partial}{\partial -\xi}\right)^n e^{-(-\xi)^2/2}$

$\hat{\wp}\varphi_n = (-1)^n\varphi_n = A_n\left(\xi - \frac{\partial}{\partial\xi}\right)^n (-1)^n e^{-(-\xi)^2/2}$

Then problem 7.9 we already did in the classroom. N=1, n+1 can prove that so we will not discuss in detail. Use relation for ladder operators above.

For problem 7.10, the average potential is compared to average kinetic energy.

$$\underbrace{\langle V \rangle}_{\text{Potential energy}} = \langle n | \hat{V} | n \rangle = ?$$

$$\underbrace{\langle T \rangle}_{\text{Kinetic energy}} = \langle n | \hat{T} | n \rangle = ?$$

$$\hat{H} = \underbrace{\frac{p^2}{2m}}_{T} + \underbrace{\frac{k^2 x^2}{2}}_{V} \quad \text{where} \quad \omega_0^2 = \frac{k}{m} \quad \text{and} \quad \beta^2 = \frac{m\omega_0}{\hbar}$$

$$x = \frac{a+a^+}{\sqrt{2}\beta} \quad \text{and} \quad p = \frac{a-a^+}{\sqrt{2}\beta}\frac{m\omega_0}{i}$$

$$\boxed{\begin{aligned} a^+ |n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ a|n\rangle &= \sqrt{n}\,|n-1\rangle \end{aligned}} \Bigg\} \text{Again, Memorize This!}$$

$$\begin{aligned}
\langle V \rangle &= \langle n | \hat{V} | n \rangle \\
&= \langle n | \tfrac{k}{2} x^2 | n \rangle \\
&= \tfrac{k}{4\beta^2} \langle n | (a + a_+)(a + a_+) | n \rangle \\
&= \tfrac{k}{4\beta^2} \Big[\langle n | (a + a_+) \sqrt{n} \, | n-1 \rangle + \langle n | (a + a_+) \sqrt{n+1} \, | n+1 \rangle \Big] \\
&= \tfrac{k}{4\beta^2} \Big[\langle n | \sqrt{n-1}\sqrt{n} \, | n-2 \rangle + \langle n | n+1 | n \rangle + \langle n | n | n \rangle + \langle n | \sqrt{n+2}\sqrt{n+1} \, | n+2 \rangle \Big] \\
&= \tfrac{k}{4\beta^2} (n+1+n) \\
&= \tfrac{k}{2\beta^2} \tfrac{\beta^2 \hbar}{m\omega_0} \tfrac{m\omega_0^2}{k} (n + \tfrac{1}{2}) \\
&= \tfrac{\hbar\omega_0}{2} (n + \tfrac{1}{2}) \\
&= \tfrac{E_n}{2}
\end{aligned}$$

Can you do the same for the <T>?

Now problem 8.35.

$\hat{H} = \frac{p^2}{2m} + \frac{k^2x^2}{2} + H(y)$

$\psi_{n1,n2} = \psi_{n1}\psi_{n2}$

$n_1 =$ quantum number of x

$n_2 =$ quantum number of y

The eigenenergies are independent of each other

(here direction independent k_x and k_y are different by the mass is same)

$$E_{n1,n2} = E_{n1} + E_{n2} = \hbar\omega_0\left(n_1 + \tfrac{1}{2}\right) + \hbar\omega_0\left(n_2 + \tfrac{1}{2}\right) = \hbar\omega_0 \underbrace{\left(n_1 + n_2 + 1\right)}_{=S+1} = \hbar\omega_0(S+1)$$

$n_1 + n_2 = S$ how many solutions exist?

$$\left.\begin{matrix} 0 & S \\ 1 & S-1 \\ 2 & S-2 \\ \vdots & \vdots \\ S & 0 \end{matrix}\right\} S+1 \text{ different stats this is order of degeneracy}$$

Ground state=0: Degeneracy(Ground state) = 1

At higher energy states, the degree of degeneracy increases.

State	Degree of Degeneracy
n	$n+1$
$\vdots$	$\vdots$
2	3
1	2
0	1

Now consider the case where the spring constants are different: $k_x=k_y+\Delta k$. The energy levels split. The oscillator external forces – they split – very simple but very useful model to explain. If the spring constants are very different: $k_x=k$ and $k_y=4k$. If you write the eigenenergy,

$$E_{nx,ny} = \hbar\omega_x\left(n_x + \tfrac{1}{2}\right) + \hbar\omega_y\left(n_y + \tfrac{1}{2}\right)$$

$$\omega_x^2 = \tfrac{k}{m} \quad \text{and} \quad \omega_y^2 = \tfrac{4k}{m} = 4\omega_x^2 \quad \Rightarrow \boxed{\omega_y = 2\omega_x}$$

$$E_{nx,ny} = \hbar\omega_x\left(n_x + \tfrac{1}{2}\right) + 2\hbar\omega_x\left(n_y + \tfrac{1}{2}\right) = \hbar\omega_x\left(n_x + 2n_y + \tfrac{3}{2}\right)$$ we proved that

If we write out we can do it the degree of degeneracy is: $E_{2,3} = \hbar\omega_x\left(\underbrace{2 + 6}_{8} + \tfrac{3}{2}\right)$

$n_x + 2n_y = 8$ for $n_x = 2, n_y = 3$

n_x	n_y	
0	4	5 States, Degree of Degeneracy
2	3	
4	2	
6	1	
8	0	

Now consider problem 9.5. The molecular and rotational Hamiltonian.

Hamiltonian:

$$\hat{H} = \frac{L^2}{2I}$$

Eigenvalue of the Hamiltonian:

$$L^2 = \hbar^2 l(l+1) \quad \text{where} \quad l = \tfrac{n}{2} \text{ an odd half-integer}$$

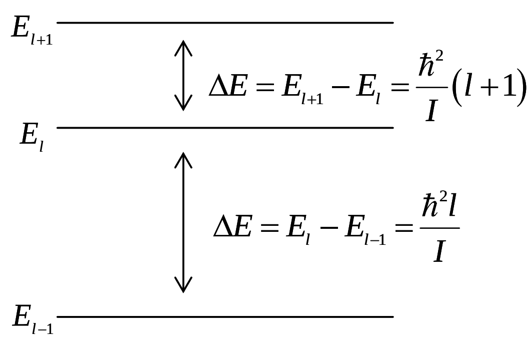
} Memorize This!

The Eigenenergy is:

$$E_l = \frac{\hbar^2 l\left(l+1\right)}{2I}$$

$$E_{l+1} = \frac{\hbar^2\left(l+1\right)\left(l+2\right)}{2I}$$

$$E_{l-1} = \frac{\hbar^2 l\left(l-1\right)}{2I}$$



$$\begin{aligned}
\Delta E = E_l - E_{l-1} &= \frac{\hbar^2 l\left(l+1\right)}{2I} - \frac{\hbar^2 l\left(l-1\right)}{2I} \\
&= \frac{\hbar^2}{2I}\left[l\left(l+1\right) - l\left(l-1\right)\right] \\
&= \frac{\hbar^2}{2I}\left[l^2 + l - l^2 + l\right] \\
&= \frac{\hbar^2}{2I}\left[2l\right] = \frac{\hbar^2 l}{I}
\end{aligned}$$

$$\begin{aligned}
\Delta E = E_{l+1} - E_l &= \frac{\hbar^2\left(l+1\right)\left(l+2\right)}{2I} - \frac{\hbar^2 l\left(l+1\right)}{2I} \\
&= \frac{\hbar^2}{2I}\left[\left(l+1\right)\left(l+2\right) - l\left(l+1\right)\right] \\
&= \frac{\hbar^2}{2I}\left[l^2 + 3l + 2 - l^2 - l\right] \\
&= \frac{\hbar^2}{2I}\left[2l + 2\right] = \frac{\hbar^2}{I}\left(l+1\right)
\end{aligned}$$