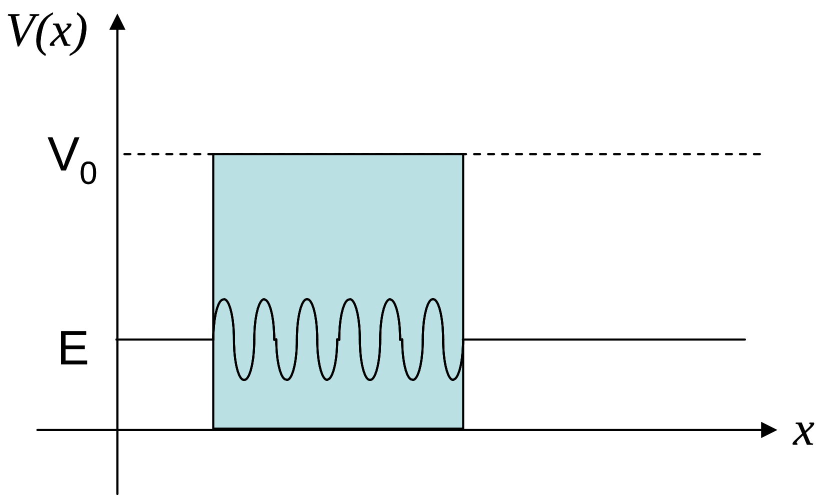
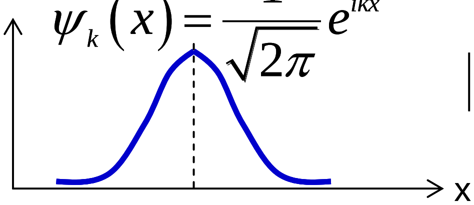
One-Dimensional Barrier Problems

Today we are studying the 1-D particle in an environment of potential barriers. In classical mechanics, if E > V_0, the particle can't go past the barrier. In quantum mechanics, the particle can go through the barrier. This is called tunneling.



We've studied the bound state of a particle in an infinite potential well, and the unbound state of a free particle.



$$\psi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

$$|\psi|^2 \xrightarrow[|x|\to 0]{} 0$$

$$N = \int_a^b |\psi|^2 dx < \infty$$

$$\int_{-\infty}^{\infty} \underbrace{|\psi|^2 dx}_{=\rho dx} \to \infty$$

Faraway, also have some probability of being. We can also consider the probabilities in small intervals ρdx. From the time independent Schrödinger equation, we got the continuity equation and the current density.

For Example:

$$\rho = 10^3 \text{ neutrons cm}^{-1} \qquad \psi = 10^{3/2} e^{i(kx-\omega t)}$$

The continuity equation.

The total time change in density is the number of particles going out.

$$\frac{\partial \overbrace{\rho}^{\text{density}}}{\partial t} + \underbrace{\nabla \cdot \vec{J}}_{\text{density current}} = 0$$

$$\int_V \frac{\partial \rho}{\partial t} dv + \int_V \nabla \cdot \vec{J} = 0 \Rightarrow \frac{\partial N}{\partial t} + \int \vec{J} \cdot ds = 0$$

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} \hat{H} \psi$$

$$\frac{\partial \psi^*}{\partial t} = \frac{i}{\hbar} \hat{H} \psi^*$$

$$\rho = \psi^* \psi \; \frac{\partial \psi^* \psi}{\partial t} = \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t} = \psi^* \left(-\frac{i}{\hbar} \hat{H} \psi \right) + \psi \left(\frac{i}{\hbar} \hat{H} \psi^* \right)$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x) = \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$$

So:

$$\frac{\partial \psi^* \psi}{\partial t} = \frac{i\hbar}{2m} \left(\psi^* \frac{\partial^2 \psi}{\partial x^2} + \psi \frac{\partial^2 \psi^*}{\partial x^2} \right) = -\frac{\partial}{\partial x} \left[\frac{\hbar}{2mi} \left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right) \right]$$

$$J_x = \frac{\hbar}{2mi} \left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right)$$

$$\vec{J} = \frac{\hbar}{2mi} \left(\psi^* \nabla \psi - \psi \nabla \psi^* \right)$$

$$T \equiv \left| \frac{J_{transmitted}}{J_{in}} \right|$$

$$R \equiv \left| \frac{J_{reflected}}{J_{in}} \right|$$

$$J_{transmitted} + J_{reflected} = 1$$

Example 1: Energy Barrier Step Function (see Figure 7.18a on page 233)

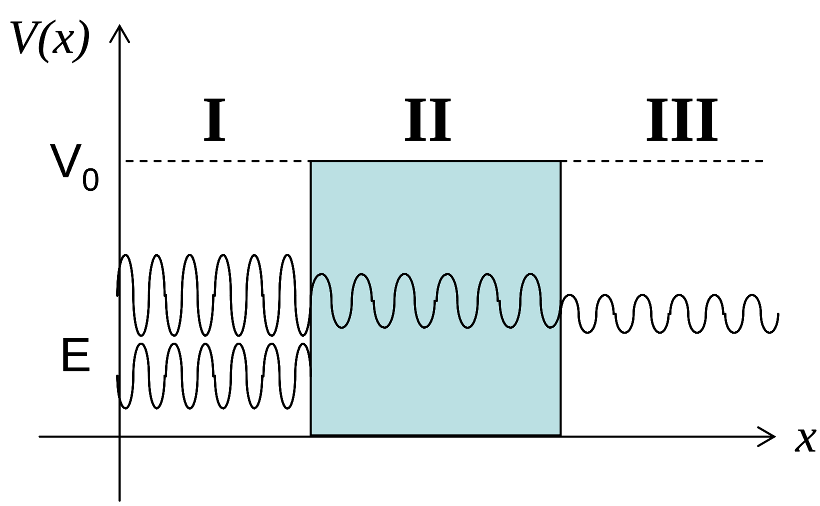
$$\psi_{in} = Ae^{i(kx-\omega t)}$$
$$\psi_{reflected} = Be^{-i(kx-\omega t)}$$
$$\psi_{transmitted} = Ce^{-i(k_2 x-\omega_2 t)}$$

We have:

$$J_{in} = \tfrac{\hbar}{2mi} 2ik_1 |A|^2$$
$$\psi_{reflected} = \tfrac{\hbar}{2mi} 2ik_1 |B|^2$$
$$\psi_{transmitted} = \tfrac{\hbar}{2mi} 2ik_2 |C|^2$$

Here:

$$T = \left|\frac{C}{A}\right|^2 \frac{k_2}{k_1}$$
$$R = \left|\frac{B}{A}\right|^2$$



Example 2: Energy Barrier Wall (see Figure 7.18b on page 233)

$$\psi_{Iin} = Ae^{i(k_1x-\omega_1t)} \qquad \frac{\hbar^2k_1^2}{2m} = E$$

$$\psi_{Ireflected} = Be^{-i(k_1x-\omega_1t)}$$

$$\psi_I = Ae^{i(k_1x-\omega_1t)} + Be^{-i(k_1x-\omega_1t)}$$

$$\psi_{Itransmitted} = Ce^{i(k_2x-\omega_2t)} \qquad \frac{\hbar^2k_2^2}{2m} = E - V > 0$$

$$\psi_{IIreflected} = De^{-i(k_2x-\omega_2t)}$$

$$\psi_{II} = Ce^{i(k_2x-\omega_2t)} + De^{-i(k_2x-\omega_2t)}$$

$$\psi_{III} = \psi_{IItransmitted} = Fe^{i(k_3x-\omega_3t)}$$

$$T = \left|\frac{F}{A}\right|^2$$

$$\psi_I(-a) = \psi_{II}(-a)$$

$$\frac{\partial}{\partial x}\psi_I(-a) = \frac{\partial}{\partial x}\psi_{II}(-a)$$

$$\psi_{II}(a) = \psi_{III}(a)$$

$$\frac{\partial}{\partial x}\psi_{II}(a) = \frac{\partial}{\partial x}\psi_{III}(a)$$

$$\overset{E>V}{} \quad \frac{1}{T} = \left|\frac{A}{F}\right|^2 = 1 + \frac{1}{4}\frac{V^2}{E(E-V)}\sin^2(2k_2a) \quad 2k_2a = n\pi \quad n = 0,1,2,\ldots$$

$$\overset{E>V \quad ik_2\to K}{} \quad \frac{1}{T} = 1 + \frac{1}{4}\frac{V^2}{E(E-V)}\underbrace{\sinh\underbrace{(2Ka)}_{\frac{\hbar^2k^2}{2m}=(V-E)}}_{=\frac{e^{2Ka}-e^{-2Ka}}{2}}$$