Chapter 6 Erenfest's Theorem Noah

The classical limit of quantum mechanics – we expect the expectation values will be classical equations.

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = i\hbar|\dot{\psi}\rangle = H|\psi\rangle \Rightarrow |\dot{\psi}\rangle = \frac{H}{i\hbar}|\psi\rangle \text{ and } \langle\dot{\psi}| = \langle\psi|\left(\frac{-H}{i\hbar}\right)$$

$$\frac{d\langle\Omega\rangle}{dt} = \frac{d}{dt}\langle\psi|\Omega|\psi\rangle = \langle\dot{\psi}|\Omega|\psi\rangle + \langle\psi|\Omega|\dot{\psi}\rangle$$

(as long as Ω doesn't depend on time, $\dot{\Omega} = 0$)

$$\frac{d\langle\Omega\rangle}{dt} = \langle\psi|\frac{-H}{i\hbar}\Omega|\psi\rangle + \langle\psi|\Omega\frac{H}{i\hbar}|\psi\rangle = \langle\psi|\frac{1}{i\hbar}(\Omega H - H\Omega)|\psi\rangle$$

$$\boxed{\frac{d\langle\Omega\rangle}{dt} = \frac{1}{i\hbar}\langle\psi|[\Omega, H]|\psi\rangle}$$ Erenhfest's Theorem

if $\Omega = \Omega(t) \Rightarrow \frac{d\langle\Omega\rangle}{dt} = \frac{1}{i\hbar}\langle\psi|[\Omega, H]|\psi\rangle + \langle\psi|\frac{d\Omega}{dt}|\psi\rangle$

Applying the theorem to variables:

$\frac{d\langle x\rangle}{dt} = \frac{1}{i\hbar}\langle\psi|[X, H]|\psi\rangle \qquad H = \frac{P_x^2}{2m} + V(x) \qquad [X, H] = \frac{1}{2m}[X, P_x^2]$

Use this relation $[A, BC] = B[A, C] + [A, B]C$

$$[X, H] = \frac{P_x}{2m}[X, P_x] + \frac{1}{2m}[X, P_x]P_x = \frac{i\hbar P_x}{m}$$

$$\langle\psi|[X, H]|\psi\rangle = \frac{1}{i\hbar}\langle\psi|\, i\hbar\frac{P_x}{2m}|\psi\rangle = \frac{1}{m}\langle\psi|P_x|\psi\rangle \Rightarrow \frac{d\langle x\rangle}{dt} = \frac{\langle p_x\rangle}{m}$$

$$\frac{d\langle P_x\rangle}{dt} = \frac{1}{i\hbar}\langle\psi|[P_x, H]|\psi\rangle = \frac{1}{i\hbar}\langle\psi|[P_x, V(x)]|\psi\rangle$$

Show that $[P_x, V(x)] = -i\hbar\frac{dV(x)}{dx}$ $\Big\{$ this is a general result, doesn't have to be a potential function

Classical Limit $\frac{d\langle P_x\rangle}{dt} = \frac{1}{i\hbar}(-i\hbar)\langle\psi|\frac{dV(x)}{dx}|\psi\rangle = -\left\langle\frac{dV(x)}{dx}\right\rangle$

This is Newton's Second Law: $\vec{F} = \dot{\vec{p}}$

We take the Hamiltonian as the same form as classical equations in the classical limit. The only difference is the operators.

Chapter 7 Simple Harmonic Oscillator

1D SHO

This one problem has many applications. Many problems can be reduced to a collection of simple harmonic oscillators. The energy of the system is the sum of the energies of all of them.

Classically: $H = \frac{p_x^2}{2m} + \frac{m\omega^2 x^2}{2}$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$x = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t + \delta_0)$$

$$\dot{x} = v_x = -C\omega\sin(\omega t + \delta_0)$$

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\omega^2 x^2 = \tfrac{1}{2}mC^2\omega^2\sin^2(\omega t+\delta_0) + \tfrac{1}{2}mC^2\omega^2\cos^2(\omega t+\delta_0)$$
$$= \tfrac{1}{2}m\omega^2 C^2$$

Energy

$E_n = (n+\tfrac{1}{2})\hbar\omega$ $\quad \hbar$ = energy · time = action = $1.055\cdot 10^{-34}$ Js $\quad \omega = \sqrt{k/m}$

Coordinate Representation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{m\omega^2 x^2}{2}\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left[E - \tfrac{1}{2}m\omega^2 x^2\right]\psi = 0$$

Substitution $x = by$ b has dimensions of length → y is dimensionless

$$\frac{d\psi}{dx} = \frac{d\psi}{dy}\frac{dy}{dx} = \frac{1}{b}\frac{d\psi}{dy} \qquad \frac{d^2\psi}{dx^2} = \frac{1}{b^2}\frac{d^2\psi}{dy^2}$$

$$\frac{1}{b^2}\frac{d^2\psi}{dy^2} + \frac{2m}{\hbar^2}\left[E - \tfrac{1}{2}m\omega^2 b^2 y^2\right]\psi = 0$$ y, ψ are dimensionless

$$\frac{d^2\psi}{dy^2} + \frac{2mEb^2}{\hbar^2}\psi - \frac{m^2\omega^2 y^2 b^4}{\hbar^2}\psi = 0$$

$b^4 = \frac{\hbar^2}{m^2\omega^2} \quad b^2 = \frac{\hbar}{m\omega} \quad b = \sqrt{\frac{\hbar}{m\omega}} \qquad \frac{2mE}{\hbar^2}\frac{\hbar}{m\omega} = \frac{2E}{\hbar\omega}$ let $\varepsilon = \frac{E}{\hbar\omega}$

$$\boxed{\frac{d^2\psi}{dy^2} + (2\varepsilon - y^2)\psi = 0}$$

Noah

$$\frac{d^2\psi}{dy^2} + (2\varepsilon - y^2)\psi = 0$$

$\boxed{1^{ST} \text{ Limit}}$ $y \to \infty$

$$\frac{d^2\psi}{dy^2} - y^2\psi = 0 \Rightarrow \psi(y) = e^{\pm y^2/2}$$

$\boxed{2^{ND} \text{ Limit}}$ $y \to 0$

$$\frac{d^2\psi}{dy^2} + 2\varepsilon\psi = 0 \Rightarrow \psi = A\cos(\sqrt{2\varepsilon}\, y) + B\sin(\sqrt{2\varepsilon}\, y)$$

$\boxed{\text{In general}}$

$$\psi(y) = u(y)e^{-y^2/2}$$

$$\frac{d\psi}{dy} = \left(-yu(y) + \frac{du(y)}{dy}\right)e^{-y^2/2}$$

$$\frac{d^2\psi}{dy^2} = e^{-y^2/2}(-y)\frac{du}{dy} - e^{-y^2/2}u - y^2ue^{-y^2/2} + \frac{d^2u}{dy^2}e^{-y^2/2} - y\frac{du}{dy}e^{-y^2/2}$$

Solve for u (substitute ~~eq~~)

$$\frac{d^2u}{dy^2}e^{-y^2/2} + \left(-2y\frac{du}{dy}\right)e^{-y^2/2} + (y^2-1)e^{-y^2/2}u + (2\varepsilon - y^2)ue^{-y^2/2} = 0$$

Combine to get:

$$\frac{d^2u}{dy^2} - 2y\frac{du}{dy} + (2\varepsilon - 1)u = 0$$

Can chose the solution to be either an even function of y OR an odd function of y

Power Series

$$u = \sum_{n=0}^{\infty} c_n y^n \qquad \begin{array}{l} n = \text{odd} \Rightarrow u = \text{odd function} \\ n = \text{even} \Rightarrow u = \text{even function} \end{array}$$

$$\frac{du}{dy} = \sum_n c_n n y^{n-1} \qquad \frac{d^2u}{dy^2} = \sum c_n n(n-1) y^{n-2}$$

Substitute into the equation:

$$\sum_n c_n n(n-1)y^{n-2} - 2\sum_n c_n n y^n + (2\varepsilon - 1)\sum_n c_n y^n = 0$$

$\therefore$ every coefficient on $y^n = 0$

$$\boxed{\sum_{n=0}^{\infty} y^n\left[c_{n+2}(n+2)(n+1)+c_n(2\varepsilon-1-2n)\right]=0}$$

$$\sum_n\left[c_n n(n-1)y^{n-2}-2c_n n y^n+(2\varepsilon-1)c_n y^n\right]=0$$

$$c_{n+2}(n+2)(n+1)-2c_n n+(2\varepsilon-1)c_n=0$$

$$c_{n+2}(n+1)(n+2)=c_n\left[2n-(2\varepsilon-1)\right]$$

$$\frac{c_{n+2}}{c_n}=\frac{2n-(2\varepsilon-1)}{(n+1)(n+2)}=\frac{2n+1-2\varepsilon}{(n+1)(n+2)}$$

this recursion relation allows us to get all the coefficients even/odd can get everything within an arbitrary constant

$$\lim_{n\to 0}\frac{c_{n+2}}{c_n}=\frac{2n}{n^2}=\frac{2}{n}$$

$$e^{y^2}=\sum_{n=0}^{\infty}\frac{y^{2n}}{n!}$$

Blows up at y^n so we have to demand that it will terminate at some point

$$\frac{c'_{n+2}}{c_n}=\frac{1}{(n/2+1)!}(n/2)!=\left(\frac{1}{n/2+1}\right)\xrightarrow[\lim n\to\infty]{}\frac{2}{n}$$

$$2n+1=2\varepsilon \qquad \varepsilon=\frac{E}{\hbar\omega}=(n+1/2)\Rightarrow E=(n+1/2)\hbar\omega)$$

Note: n can be as low as zero, the energy is quantized to be a solution of Schrödinger's equation
u is a polynomial of degree n.

$$\boxed{\frac{d^2u}{dy^2}-2y\frac{du}{dy}+2nu=0}$$ Hermite Polynomial

We can use the recursion relation $\frac{c_{n+2}}{c_n}$ to get $(1-2\varepsilon)$:

$$n=0:\ \frac{c_2}{c_0}=\frac{2n+1-2\varepsilon}{(n+1)(n+2)}=\frac{1-2\varepsilon}{2}\Rightarrow\tfrac{1}{2}(1-2\varepsilon)c_0=c_2$$

$$c_4=\tfrac{1}{12}(5-2\varepsilon)c_2 \qquad c_3=\tfrac{1}{2}(1-\tfrac{2}{3}\varepsilon)c_1$$

Even y $\quad u_e=c_0\left[1+\tfrac{1}{2}(1-2\varepsilon)y^2+\tfrac{1}{2}(1-2\varepsilon)\frac{(5-2\varepsilon)}{12}y^4+\ldots\right]$

Odd y $\quad u_o=c_1\left[y+\left(\frac{3-2\varepsilon}{6}\right)y^3+\ldots\right]$

Subject to Boundary Conditions

$u_e(0)=1\Rightarrow c_0=1$ $\qquad u_o(0)=0$ $\qquad \frac{E}{\hbar\omega}=\varepsilon=(n+1/2)$

$u_e'(0)=0$ $\qquad u_o'(0)=1\Rightarrow c_1=1$

Hermite Polynomials

$H_0(y) = 1$ $\qquad$ $H_1(y) = 2y$

$H_2(y) = -2(1-2y^2)$

$H_4(y) = 12(1-4y^2+\frac{4}{3}y^4)$

etc.

Once you have the Hermite Polynomials ..

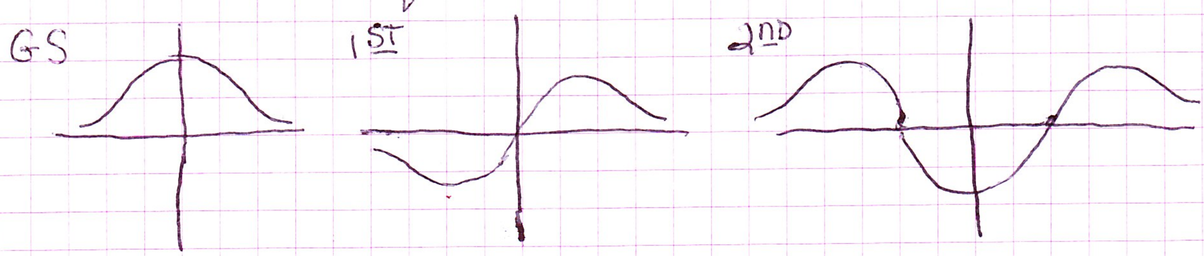
$\psi_n(y) = C_n H_n(y)\, e^{-y^2/2}$

where $y = bx = \underbrace{\sqrt{\frac{m\omega}{\hbar}}}_{\text{dimension } 1/\text{length}}\, x$ $\qquad$ $\sqrt{\frac{m\omega}{\hbar}} = \sqrt{\frac{MT^{-1}}{LMLT^{-1}}} = \sqrt{\frac{1}{L^2}} = \frac{1}{L}$

in terms of x:

$$\psi_n(x) = C_n \underbrace{H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)}_{\text{Hermite}} \underbrace{e^{-(m\omega/2\hbar)x^2}}_{\text{Gaussian}}$$

n is the number of nodes

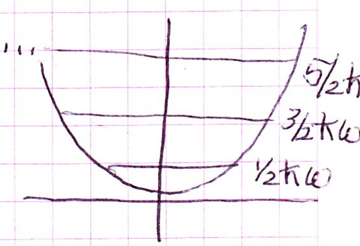


Dirac Notation

$H = \frac{P_x^2}{2m} + \frac{1}{2}m\omega^2 X^2$ $\qquad$ $E_n = (n+\tfrac{1}{2})\hbar\omega$ $\quad$ $n = 0, 1, 2, \ldots$

$\langle\psi|H|\psi\rangle = \frac{1}{2m}\langle\psi|P_x^2|\psi\rangle + \frac{1}{2}m\omega^2\langle\psi|X^2|\psi\rangle$

$\langle\psi|P_x^2|\psi\rangle = \langle\psi|P_x\, P_x|\psi\rangle = \langle\phi|\phi\rangle > 0$



$\varepsilon = \frac{E}{\hbar\omega} = (n+\tfrac{1}{2})$

Hermite Polynomials - Math Tricks

$$\frac{d^2 H_n(y)}{dy^2} - 2y\frac{dH_n(y)}{dy} + 2n H_n(y) = 0$$

for every n there is only one polynomial solution

Stuff you can easily derive by looking at this differential equation:

$$\frac{d^2}{dy^2}\left(\frac{dH_n}{dy}\right) - 2y\frac{d^2H_n}{dy} - \frac{2dH_n}{dy} + \frac{2ndH_n}{dy} = 0$$

$$\frac{d^2}{dy^2}\frac{dH_n}{dy} - 2y\frac{d^2H_n}{dy^2} + 2(n-1)\frac{dH_n}{dy} = 0$$

$$\Rightarrow \frac{dH_n}{dy} = b\,H_{n-1} \qquad H_n(y) = \_\_ + 2^n y^n \qquad dH_n/dy = \_\_ + 2^n n y^{n-1}$$

$$H_{n-1} = \cdots + 2^{n-1} y^n \qquad b\,dH_n/dy = \_\_ + n2^{n}by^{n-1}$$

$$2^n n = b2^{n-1}$$
$$b = n2$$

$$\frac{dH_n}{dy} = 2n H_{n-1} \quad \text{recursion relation}$$

$$F(S,y) = \sum_{n=0}^{\infty} \frac{H_n(y)S^n}{n!} \quad \text{generating function}$$

$$\frac{\partial F(S,y)}{\partial y} = \sum_{n=0}^{\infty}\frac{(dH_n/dy)S^n}{n!} = \sum_{n=1}^{\infty}\frac{H_{n-1}S^{n-1}}{n!} = 2S\sum_{n=1}^{\infty}\frac{H_{n-1}S^{n-1}}{(n-1)!} = 2S\sum_{m=0}^{\infty}\frac{H_m S^m}{m!}$$

$m = n-1$

$$\frac{\partial F(S,y)}{\partial y} = 2sF(S,y) \Rightarrow \frac{\partial F}{F} = 2s\,\partial y$$

$$\ln F(S,y) = 2sy + C \Rightarrow F = Ce^{2sy}$$

$$F(s,0) = C \qquad F(S,y) = F(s,0)e^{2sy} \qquad F(s,0) = \sum_{n=1}^{\infty}\frac{H_n(0)s^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^{n/2}s^n}{(n/2)!}$$

$$H_n(y) = (\ldots)v_e(y)$$
$$H_n(0) = (\ldots)v_e(0)$$

$v_e(0) = 1 \leftarrow$ put $n = 2m$

$$F(s,0) = \sum_{m=0}^{\infty}\frac{(-1)^m s^{2m}}{m!} \qquad e^{-s^2} = \sum_{m=0}^{\infty}\frac{(-s^2)^m}{m!}$$

only have to use this for a generating function

$$F(s,y) = e^{-s^2}e^{2sy} = e^{-s^2+2sy} = \underbrace{\sum_{n=0}^{\infty}\frac{H_n(y)s^n}{n!}}$$

this is how Mertzbacher defines the Hermite polynomial (not Shankar's)

Noah

$\psi_n(x) = C_n H_n(y) e^{-y^2/2} \qquad y = \sqrt{\frac{m\omega}{\hbar}}\, x \qquad x = \sqrt{\frac{\hbar}{m\omega}}\, y \qquad dx = \sqrt{\frac{\hbar}{m\omega}}\, dy$

$$\int_{-\infty}^{\infty} \psi^* \psi \, dx = \int_{-\infty}^{\infty} \psi_n^*(y)\, \psi_n(y) \sqrt{\frac{\hbar}{m\omega}}\, dy = \sqrt{\frac{\hbar}{m\omega}}\, |C_n|^2 \int_{-\infty}^{\infty} [H_n(y)]^2 e^{-y^2} dy = \sqrt{\frac{\hbar}{m\omega}}\, |C_n|^2\, 2^n n! \sqrt{\pi} = 1$$

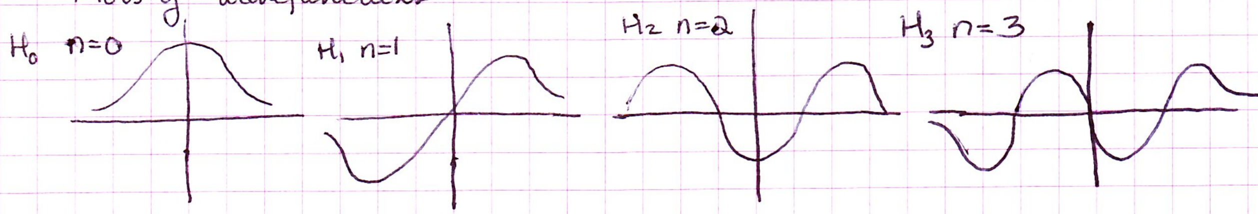
$|C_n|^2 = \sqrt{\frac{m\omega}{\pi\hbar}}\, \frac{1}{2^n n!} \qquad C_n = \frac{1}{2^{n/2}\sqrt{n!}} \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}$ within an arbitrary phase factor

$\psi_n(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \frac{1}{2^{n/2}\sqrt{n!}}\, H_n(y)\, e^{-y^2/2} \qquad y = \sqrt{\frac{m\omega}{\hbar}}\, x$

all the roots are real of H_n, a polynomial of degree n has n solutions/roots

$H_0(y) = 1$ — no zeros
$H_1(y) = 2y$ — 1 zero @ $y=0$
$H_2(y) = -2(1-2y^2)$ — 2 zeros @ $y = \pm\sqrt{1/2}$
$\vdots$

Property of all eigenfunctions: more modes → kinetic energy expectation increases
Plots of wavefunctions



Solving the Harmonic Oscillator w/ Commutation Relations

$H|E\rangle = E|E\rangle \qquad H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2 \qquad [X,P] = i\hbar \mathbb{I}$

$a = \sqrt{\frac{m\omega}{2\hbar}}\, X + \frac{i}{\sqrt{2\hbar m\omega}}\, P_x \qquad \sqrt{2\hbar m\omega} \Rightarrow \sqrt{(MLT)^{-1} LMT}$ = dimensionless

$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\, X - \frac{i}{\sqrt{2\hbar m\omega}}\, P_x \qquad \sqrt{\frac{m\omega}{2\hbar}} \Rightarrow \sqrt{\frac{MT^{-1}}{2MLT^{-1}L}} = \sqrt{\frac{1}{L^2}} = \frac{1}{L}$ = dimensions of inverse length

$$[a,a^\dagger] = \sqrt{\frac{m\omega}{2\hbar}}\,(-i)\,\frac{1}{\sqrt{2\hbar m\omega}}\,[X,P_x] + \frac{i}{\sqrt{2\hbar m\omega}}\sqrt{\frac{m\omega}{2\hbar}}\,[P_x,X] = \frac{-i}{2\hbar}[i\hbar] + \frac{i}{2\hbar}[-i\hbar] = 1$$

$[a,a] = 0$

$[a^\dagger,a^\dagger] = 0$

$[a,a^\dagger] = 1$

define: $(a^\dagger a)^\dagger = a^\dagger a$ — self-adjoint, so eigenvalues are real and turn out to be positive integers

Ladder Operators

$a = \sqrt{\frac{m\omega}{2\hbar}}\, X + i\sqrt{\frac{1}{2\hbar m\omega}}\, P_x \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\, X - i\sqrt{\frac{1}{2m\omega\hbar}}\, P_x$ $\quad a, a^\dagger$ are dimensionless

invert these to get:

$$X = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a + a^\dagger) \quad P_x = i\left(\frac{\hbar m\omega}{2}\right)^{1/2}(a^\dagger - a)$$

$$H = \frac{P_x^2}{2m} + \frac{1}{2}m\omega^2 X^2 = \hbar\omega(a^\dagger a + 1/2)$$ $\therefore$ $a^\dagger a$ are also eigenvalues of H

$a^\dagger a|n\rangle = n|n\rangle = N|n\rangle \quad a^\dagger a = N =$ number operator

$[a, N] = [a, a^\dagger a] = [a, a^\dagger]a + a^\dagger[a, a] = a$

$N(a|n\rangle) = [N, a]|n\rangle = aN|n\rangle = -a|n\rangle + na|n\rangle = (n-1)a|n\rangle$

$[N, a] = -a$

Ladder operators can't operate indefinitely

$a|n_0\rangle = |0\rangle = 0|n_0\rangle$

$a^\dagger a|n_0\rangle = 0|n_0\rangle = |0\rangle \qquad a^\dagger a|0\rangle = |null\rangle$

$[N, a^\dagger] = a^\dagger$

$N|(a^\dagger|n\rangle) = [N, a^\dagger]|n\rangle + a^\dagger N|n\rangle = a^\dagger|n\rangle + a^\dagger n|n\rangle = (n+1)a^\dagger|n\rangle$

$\therefore \left.\begin{array}{l} a|n\rangle = c_-|n-1\rangle \\ a^\dagger|n\rangle = c_+|n+1\rangle \end{array}\right\}$ can find the constants easily, take the adjoint

$\langle n|a^\dagger a|n\rangle = \langle n|N|n\rangle = n\langle n|n\rangle = n \qquad \langle n-1|c_-^* c_-|n-1\rangle = |c_-|^2\langle n|n\rangle = |c_-|^2 = n \Rightarrow c_- = \sqrt{n}$

$\langle n|aa^\dagger|n\rangle = \langle n|[a, a^\dagger]|n\rangle + \langle n|a^\dagger a|n\rangle = 1 + n\langle n|n\rangle = n+1$

$\langle n+1|n+1\rangle|c_+|^2 = n+1 = |c_+|^2 \qquad c_+ = \sqrt{n+1}$

once we know c_- and c_+ this is a very useful relation

$a^\dagger a|0\rangle = 0|0\rangle$

$a^\dagger|0\rangle = \sqrt{1}|1\rangle$

$a^{\dagger 2}|0\rangle = a^\dagger|1\rangle = \sqrt{2}|2\rangle \quad \Rightarrow |2\rangle = \frac{1}{\sqrt{2}}(a^\dagger)^2|0\rangle \qquad |0\rangle = \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$

$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\, X - i\sqrt{\frac{1}{2m\omega\hbar}}\, P_x \longrightarrow \frac{1}{2}\left(y - \frac{d}{dy}\right)$

$a = \sqrt{\frac{m\omega}{2\hbar}}\, X + i\sqrt{\frac{1}{2m\omega\hbar}}\, P_x \longrightarrow \frac{1}{\sqrt{2}}\left(y + d/dy\right)$

Noah

$$\langle x|n\rangle = \frac{1}{\sqrt{n!}}\langle x|(a^+)^n|0\rangle$$

$$\psi_n(y) = \frac{1}{\sqrt{n!}}\frac{1}{2^{n/2}}\left(y - \frac{d}{dy}\right)^n\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-y^2/2}$$

$$\psi_n(y) = \frac{1}{\sqrt{n!}}\frac{1}{2^{n/2}}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4} H_n(y)\, e^{-y^2/2}$$

$$\psi_n(y) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\frac{1}{2^{n/2}}\frac{1}{\sqrt{n!}}\, e^{-y^2/2}\underbrace{\left[e^{+y^2/2}\left(y - \frac{d}{dy}\right)^n e^{-y^2/2}\right]}$$

very convenient definition
polynomial - original

It is much easier to solve SHO using operator algebra than solving differential equations. As energy ↑ the number of nodes ↑

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

take the x-basis vectors in terms of y

$\langle y|a|n\rangle = \sqrt{n}\,\langle y|n-1\rangle$

$\frac{1}{\sqrt{2}}(y + d/dy)\,\psi_n(y) = \sqrt{n}\,\psi_{n-1}(y)$

$1/\sqrt{2}\,(y + d/dy)\,(m\omega/\hbar\pi)^{1/4}(1/2^{n/2})(1/\sqrt{n!})\,H_n(y)\,e^{-y^2} = \sqrt{2}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\frac{1}{2^{(n-1)/2}}\frac{1}{\sqrt{(n-1)!}}\,H_{n-1}(y)\,e^{-y^2/2}$

$\Rightarrow \dfrac{dH_n(y)}{dy} = 2n\,H_{n-1}(y)$ recursion relation

start with $H_0(y) = 1$

$\dfrac{dH_1}{dy} = 2H_0(y) = 2 \qquad H_1 = 2y + C \quad C = 0$ because has to be odd

$\dfrac{dH_2}{dy} = 2\cdot 2\cdot 2y = 8y \Rightarrow H_2 = \dfrac{8y^2}{2} + C = 4y^2 + C$ (can't say $C = 0$ for the even functions)

Normalization: $H_n(0) = \dfrac{n!\,(-1)^{n/2}}{(n/2)!}$ for even n

all real functions $\quad C = \dfrac{2!(-1)^1}{1!} = -2$

$H_2 = 4y^2 - 2 \qquad H_3 = \left(\dfrac{4y^3}{3} - 2y\right)(2\cdot 3) = 8y^3 - 12y + C \qquad C = 0$ for odd functions

Not Hermitian operators $\quad a \to \frac{1}{\sqrt{2}}(y + d/dy) \quad a^\dagger \to \frac{1}{\sqrt{2}}(y - d/dy) \quad (a+a^\dagger) = \sqrt{2}\, y$

$$\langle n|X|n'\rangle = \langle n|\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)|n'\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle n|a|n'\rangle + \langle n|a^\dagger|n'\rangle\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,\delta_{n,n'-1} + \sqrt{n'+1}\,\delta_{n,n'+1}\right]$$

$$X = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\begin{bmatrix} 0 & \sqrt{1} & & & \\ \sqrt{1} & 0 & \sqrt{2} & & \\ & \sqrt{2} & 0 & \sqrt{3} & \\ & & \sqrt{3} & \ddots & \end{bmatrix}$$

$$\langle n|P_x|n'\rangle = i\left(\frac{m\omega\hbar}{2}\right)^{1/2}\langle n|a^\dagger - a|n'\rangle = \left(\sqrt{n'+1}\,\delta_{n,n'+1} - \sqrt{n'}\,\delta_{n,n'-1}\right) i\left(\frac{m\omega\hbar}{2}\right)^{1/2}$$

$\boxed{\text{Using the Operators}}$

$$a^\dagger|2\rangle = \sqrt{3}|3\rangle \qquad |3\rangle = \frac{a^\dagger}{\sqrt{3}}|2\rangle = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{2}}(a^\dagger)^3|0\rangle = \frac{1}{\sqrt{3}}(a^\dagger)^3|0\rangle$$

$$\Rightarrow |n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \qquad H|n\rangle = \hbar\omega(n+1/2)|0\rangle$$

also can do:

$$a = \sqrt{\frac{m\omega}{2\hbar}}X + i\sqrt{\frac{1}{2m\omega\hbar}}P_x \qquad a|0\rangle = |null\rangle$$

$$\langle x|\sqrt{\frac{m\omega}{2\hbar}}X + i\sqrt{\frac{1}{2m\omega\hbar}}P_x|0\rangle = \langle x|null\rangle = 0$$

$$\langle \quad = \sqrt{\frac{m\omega}{2\hbar}}\,x\langle x|0\rangle + i\sqrt{\frac{1}{2m\omega\hbar}}\langle x|P_x|0\rangle = 0$$

but $\langle x|P_x|0\rangle = \int dx' \underbrace{\langle x|P_x|x'\rangle}_{-i\hbar\delta(x-x')}\underbrace{\langle x'|0\rangle}_{\psi_0(x')} = -i\hbar\frac{\partial\psi_0}{\partial x}$

$$\sqrt{\frac{m\omega}{2\hbar}}\,x\langle x|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\,x\psi_0(x)$$

$$\therefore \sqrt{\frac{m\omega}{2\hbar}}\,x\psi_0(x) + i\sqrt{\frac{1}{2m\omega\hbar}}\left(-i\hbar\frac{\partial\psi_0}{\partial x}\right) = 0 \quad \text{let } y = \sqrt{\frac{m\omega}{\hbar}}\,x$$

$$\frac{d\psi_0}{dy} = \frac{d\psi_0}{dx}\frac{dx}{dy} = \frac{d\psi_0}{dx}\left(\sqrt{\frac{\hbar}{m\omega}}\right) \Rightarrow \frac{1}{\sqrt{2}}y\psi_0(y) + \frac{1}{\sqrt{2}}\frac{d\psi_0(y)}{dy} = 0 \Rightarrow \psi_0 = C_0 e^{-y^2/2}$$

$$\psi_0 = C_0 e^{-m\omega x^2/2\hbar}$$

$$\int \psi_0^*\psi_0\,dx = 1 \Rightarrow |C_0|^2 = \int_{-\infty}^{\infty} e^{-m\omega x^2/\hbar}dx \Rightarrow C_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \Leftarrow \text{normalization Constant}$$

Using the Ladder Operators To Find Expectation Values!

$$X^2 = \left[\sqrt{\frac{\hbar}{2m\omega}}\,(a+a^\dagger)\right]^2 = \frac{\hbar}{2m\omega}\left[aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger\right]$$

$\langle n | X^2 | n' \rangle = \ldots$

$n' = n$: find terms with equal numbers of $a^\dagger$ and a, then you can figure it out

$\langle n | X^4 | n \rangle = \ldots$

$$H = \frac{P_x^2}{2m} + \frac{1}{2} m\omega^2 X^2 = \frac{1}{2m}\left(i\sqrt{\frac{m\omega\hbar}{2}}\,(a^\dagger - a)\right)^2 + \frac{1}{2} m\omega^2 \frac{\hbar}{2m\omega}(a^\dagger + a)^2$$

$$H = \frac{-1}{2m}\frac{m\omega\hbar}{2}(-aa^\dagger - a^\dagger a + a^\dagger a^\dagger + aa) + \frac{1}{2} m\omega^2 \frac{\hbar}{2m\omega}(a^\dagger a^\dagger + aa^\dagger + a^\dagger a^\dagger + aa)$$

$$H = \frac{\omega\hbar}{4}(aa^\dagger + a^\dagger a - a^\dagger a^\dagger - aa) + \frac{\hbar\omega}{4}(a^\dagger a + aa^\dagger + a^\dagger a^\dagger + aa)$$

$$H = \frac{\hbar\omega}{2}(aa^\dagger + a^\dagger a)$$

$$\langle n | \frac{P_x^2}{2m} | n \rangle = \frac{1}{4}\hbar\omega \langle n | 2a^\dagger a + 1 | n \rangle = \hbar\omega\left(\frac{2n+1}{4}\right) = \frac{\hbar\omega}{2}(n + 1/2)$$

$$\langle n | \tfrac{1}{2} m\omega^2 X | n \rangle = \frac{1}{2} m\omega^2 \langle n | 2a^\dagger a + 1 | n \rangle = 1/2\, m\omega^2 = \frac{\hbar\omega}{2}(n + 1/2)$$

energy is shared equally between K.E. and P.E for SHO

Chapter 8 Path Integrals

Skip for now...

Chapter 9 The Heisenberg Uncertainty Relations

Derivation (10-23-2008)

$[\Omega, \Lambda] = i\Gamma$

If Ω and Λ are Hermitian, then Γ is Hermitian.

$(\Delta\Omega)(\Delta\Lambda) \geq \frac{1}{2}|\langle\Gamma\rangle|$ uncertainty

$(\Delta\Omega)^2 = \langle\Psi|(\Omega - \langle\Omega\rangle)^2|\Psi\rangle$ = depends on the state

$(\Delta\Lambda)^2 = \langle\Psi|(\Lambda - \langle\Lambda\rangle)^2|\Psi\rangle$

Lets define $\hat{\Omega} = \Omega - \langle\Omega\rangle$, $\hat{\Lambda} = \Lambda - \langle\Lambda\rangle$ also Hermitian

$(\Delta\Omega)^2 = \langle\Psi|(\hat{\Omega})^2|\Psi\rangle = \langle\Psi|\hat{\Omega}^+\hat{\Omega}|\Psi\rangle = \langle\hat{\Omega}\Psi|\hat{\Omega}\Psi\rangle \geq 0$ by definition of magnitude

likewise

$(\Delta\Lambda)^2 = \langle\Psi|(\hat{\Lambda})^2|\Psi\rangle = \langle\hat{\Lambda}\Psi|\hat{\Lambda}\Psi\rangle \geq 0$

$(\Delta\Omega)^2(\Delta\Lambda)^2 = \langle\hat{\Omega}\Psi|\hat{\Omega}\Psi\rangle\langle\hat{\Lambda}\Psi|\hat{\Lambda}\Psi\rangle$

$|u|^2|v|^2 \geq |\vec{u}\cdot\vec{v}|^2$ equality only if $\vec{u} = c\vec{v}$

$\langle u|u\rangle\langle v|v\rangle \geq |\langle u|v\rangle|^2$: Schwartz Inequality

$\therefore\ (\Delta\Omega)^2(\Delta\Lambda)^2 \geq |\langle\hat{\Omega}\Psi|\hat{\Lambda}\Psi\rangle|^2$

$\langle\hat{\Omega}\Psi|\hat{\Lambda}\Psi\rangle = \langle\Psi|\hat{\Omega}^+\hat{\Lambda}|\Psi\rangle = \langle\Psi|\hat{\Omega}\hat{\Lambda}|\Psi\rangle = \frac{1}{2}\langle\Psi|\hat{\Omega}\hat{\Lambda} + \hat{\Lambda}\hat{\Omega}|\Psi\rangle + \frac{1}{2}\langle\Psi|[\hat{\Omega},\hat{\Lambda}]|\Psi\rangle$

$[\hat{\Omega},\hat{\Lambda}] = [\hat{\Omega} - \langle\Omega\rangle, \hat{\Lambda} - \langle\Lambda\rangle] = [\Omega, \Lambda] = i\Gamma$

$\therefore\ \langle\Psi|\hat{\Omega},\hat{\Lambda}|\Psi\rangle = \frac{1}{2}\langle\Psi|\hat{\Omega}\hat{\Lambda} + \hat{\Lambda}\hat{\Omega}|\Psi\rangle + \frac{1}{2}i\underbrace{\langle\Psi|\Gamma|\Psi\rangle}_{a+ib}$ $|a+ib|^2 = a^2+b^2$

$(\Delta\Omega)^2(\Delta\Lambda)^2 \geq |\langle\hat{\Omega}\Psi|\hat{\Lambda}\Psi\rangle|^2 = \left|\frac{1}{2}\langle\Psi|\hat{\Omega}\hat{\Lambda} + \hat{\Lambda}\hat{\Omega}|\Psi\rangle + \frac{i}{2}\langle\Psi|\Gamma|\Psi\rangle\right|^2 = \frac{1}{4}\left[|\langle\Psi|\hat{\Omega}\hat{\Lambda} + \hat{\Lambda}\hat{\Omega}|\Psi\rangle|^2 + |\langle\Psi|\Gamma|\Psi\rangle|^2\right]$

$(\Delta\Omega)^2(\Delta\Lambda)^2 \geq \frac{1}{4}|\langle\Psi|\Gamma|\Psi\rangle|^2$

$\therefore\ \Delta\Omega\,\Delta\Lambda \geq \frac{1}{2}\langle\Psi|\Gamma|\Psi\rangle$ generalized uncertainty principle of Heisenberg

The equality sign will hold with these two conditions

① $(\Omega - \langle\Omega\rangle)|\Psi\rangle = c(\Lambda - \langle\Lambda\rangle)|\Psi\rangle$

② $\langle\Psi|\hat{\Omega}\hat{\Lambda} + \hat{\Lambda}\hat{\Omega}|\Psi\rangle = 0$

Consider $\Omega = P_x$, $\Lambda = X$ $[\Omega, \Lambda] = [P_x, X] = i\hbar I$

$(\Delta P_x)^2(\Delta X)^2 \geq \frac{1}{4}|\langle\Psi|\Gamma|\Psi\rangle|^2 = \frac{1}{4}|i\hbar|^2$

$\Delta P_x\,\Delta X \geq \hbar/2$ Now we have proven Heisenberg's uncertainty principles from the postulates of Quantum Mechanics

Noah

① $(P_x - \langle P_x \rangle)|\Psi\rangle = c(x - \langle x \rangle)|\Psi\rangle$ The basis bra vectors are ~~the~~ eigenvectors of this operator $\langle x|$

② $\langle \Psi | (P_x - \langle P_x \rangle)(x - \langle x \rangle) + (x - \langle x \rangle)(P_x - \langle P_x \rangle) | \Psi \rangle = 0$

$-i\hbar \frac{d\Psi(x)}{dx} - \langle P_x \rangle \Psi(x) = c(x - \langle x \rangle)\Psi(x)$

$\Psi(x)$ will satisfy the equality sign $\Psi(x) = \langle x | \Psi \rangle$