Homework Set # 2   95.338/538

Please write legibly and show the steps

1. Which of the following waves correspond to traveling waves? What are their velocities given a, b and c are positive constants.

$\phi(z,t) = (az-bt)^2$

$\phi(z,t) = (az+bt+ct)^2$

$\phi(z,t) = 1/(az^2+b)$

2. Problem 1.4 $\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$

$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$

3. Problem 1.6

4. An electromagnetic wave in SI units is represented by $E_y = 2\cos[2\pi\, 10^{14}(t-x/c)]$. What are the frequency, wavelength, direction of propagation, velocity and amplitude of the wave?

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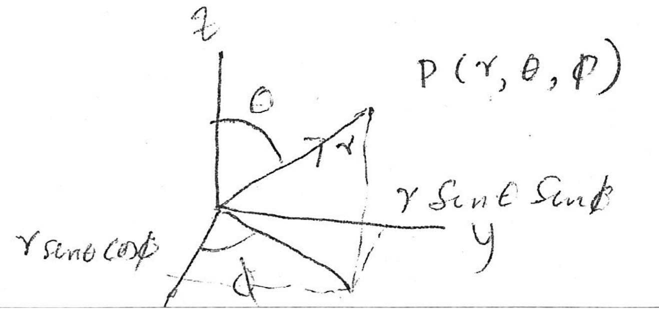

Homework Set #2 95.538 12Feb09 Meg Noah $\frac{19}{20}$

1) Which of the following waves correspond to traveling waves? What are their velocities given a, b, and c are positive constants.

$\phi(z,t) = (az - bt)^2$ traveling $v = b/a$

$\phi(z,t) = (az + (b+c)t)^2$ traveling $v = -\frac{(b+c)}{a}$

$\phi(z,t) = 1/(az^2 + b)$ not traveling, not a function of time. $v=0$

I'm not sure what steps to show. Generally $(kz - \omega t) \Rightarrow v = \omega/k$

Solving the wave equation

$$\nabla^2\phi = \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}$$

$$\frac{\partial^2}{\partial z^2}(az-bt)^2 = \frac{1}{v^2}\frac{\partial^2(az-bt)^2}{\partial t^2}$$

$$2a^2 = \frac{1}{v^2}2b^2$$

$$v = b/a$$

$$\nabla^2\phi = \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}$$

$$\frac{\partial^2}{\partial z^2}(az+(b+c)t)^2 = \frac{1}{v^2}\frac{\partial^2(az+(b+c)t)^2}{\partial t^2}$$

$$2a^2 = \frac{1}{v^2}\left(2(b+c)^2\right)$$

$$v = \frac{(b+c)}{a}$$

2) Problem 1.4: Prove that the spherical harmonic function $\frac{1}{r}e^{i(kr-\omega t)}$ is a solution of the 3-D wave equation, where $r = (x^2+y^2+z^2)^{1/2}$.

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{e^{i(kr-\omega t)}}{r}\right)\right]$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r} r^2\left[\frac{-e^{i(kr-\omega t)}}{r^2} + \frac{ike^{i(kr-\omega t)}}{r}\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left[-e^{i(kr-\omega t)} + ikre^{i(kr-\omega t)}\right]$$

$$\nabla^2 f = \frac{1}{r^2}\left[-ike^{i(kr-\omega t)} + ike^{i(kr-\omega t)} - k^2re^{i(kr-\omega t)}\right] = -k^2\left[\frac{e^{i(kr-\omega t)}}{r}\right] = -k^2 f$$

$$\nabla^2 f = -k^2 f$$

$$\frac{1}{u^2}\frac{\partial^2 f}{\partial t^2} = \frac{1}{u^2}\frac{\partial}{\partial t}\left(\frac{1}{r}(-i\omega)e^{i(kr-\omega t)}\right) = \frac{1}{u^2}(-i\omega)^2\frac{1}{r}e^{i(kr-\omega t)} = -\frac{1}{u^2}\omega^2 f = -k^2 f$$

$$\nabla^2 f = \frac{1}{u^2}\frac{\partial^2 f}{\partial t^2} = -k^2 f$$

3) Problem 1.6: Derive the formulas $u_g = u - \lambda\, du/d\lambda$

$$u_g = \frac{d\omega}{dk} = \frac{d(ku)}{dk} = u + k\frac{du}{dk} = u + k\frac{du}{d\lambda}\underbrace{\frac{d\lambda}{dk}} = u - \frac{2\pi}{k}\frac{du}{d\lambda} = u - \lambda\frac{du}{dk}$$

$$\lambda = \frac{2\pi}{k} \Rightarrow \frac{d\lambda}{dk} = -\frac{2\pi}{k^2}$$

NOTE BOOK

Problem 1.6 continued... derive $\frac{1}{u_g} = \frac{1}{u} - \frac{\lambda_o}{c}\frac{dn}{d\lambda_o}$

$$\frac{1}{u_g} = \frac{dk}{d\omega} = \frac{d}{d\omega}\left(\frac{n\omega}{c}\right) = \frac{n}{c} + \frac{\omega}{c}\frac{dn}{d\omega} = \frac{1}{u} + \frac{\omega}{c}\frac{dn}{d\lambda_o}\frac{d\lambda_o}{d\omega}$$

$$\omega = ku = \frac{kc}{n} \Rightarrow \frac{\omega}{c} = \frac{k}{n} \qquad \lambda_o = \frac{2\pi}{k} = \frac{2\pi c}{n\omega} \Rightarrow \frac{d\lambda_o}{d\omega} = -\frac{2\pi c}{n\omega^2}$$

$$\frac{\omega}{c}\frac{d\lambda_o}{d\omega} = \frac{\omega}{c}\left(\frac{-2\pi c}{n\omega^2}\right) = -\frac{2\pi}{n\omega} = -\frac{2\pi}{n}\left(\frac{n}{kc}\right) = \frac{-\lambda_o}{c}$$

$$\frac{1}{u_g} = \frac{1}{u} + \frac{\omega}{c}\frac{dn}{d\lambda_o}\frac{d\lambda_o}{d\omega} = \frac{1}{u} - \frac{\lambda_o}{c}\frac{dn}{d\lambda_o}$$

4) An electromagnetic wave in SI units is represented by $E_y = 2\cos[2\pi 10^{14}(t - \frac{x}{c})]$. What are the frequency, wavelength, direction of propagation, velocity, and amplitude of the wave?

$$E_y = 2\cos[2\pi 10^{14}(t - x/c)] = 2\cos\left[-\left(\frac{2\pi 10^{14} x}{c} - 2\pi 10^{14} t\right)\right]$$

$$E_y = 2\cos\left[\frac{2\pi 10^{14} x}{c} - 2\pi 10^{14} t\right] = A\cos(kx - \omega t)$$

$$\omega = 2\pi 10^{14}\ \frac{rad}{s} \qquad f = \frac{\omega}{2\pi} = 10^{14}\ Hz$$

$$k = \frac{2\pi 10^{14}}{c} \qquad \lambda = \frac{2\pi}{k} = \left(\frac{c}{10^{14}}\right) m$$ ← 3 microns

direction of propagation: $+y$ direction

+x direction

$$v = \frac{\omega}{k} = c = 3 \cdot 10^8\ ms^{-1}$$

Amplitude = 2m V/m

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