Meg A. Noah

2. Flow Kinematics & Rotation.

The following equation describes a 2-D potential flow: $\phi(\rho,\theta) = -\frac{\Gamma}{2\pi}\theta$

a) Calculate the velocity field by taking the gradient of the potential

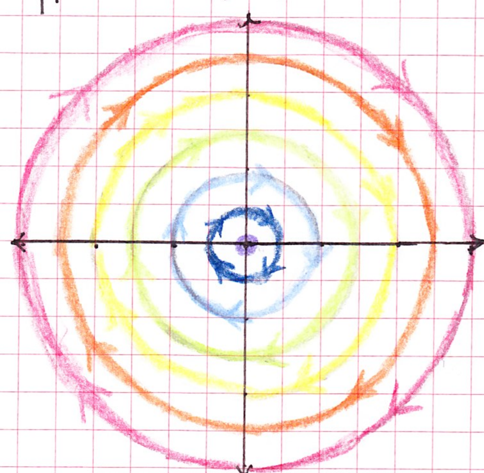
$$\vec{u}(\rho,\theta) = \hat{\rho}\frac{\partial\phi}{\partial\rho} + \hat{\theta}\frac{1}{\rho}\frac{\partial\phi}{\partial\theta} = \frac{-\Gamma}{2\pi\rho}\hat{\theta}$$

b) Draw the streamlines for the flow.
(reference p. 287 "Fundamentals of Fluid Mechanics" Munson, Young, Okiish, Herbsch

$$u_\theta = -\frac{\partial\Psi}{\partial\rho} = \frac{-\Gamma}{2\pi\rho} \Rightarrow \partial\Psi = \frac{\Gamma}{2\pi}\frac{d\rho}{\rho} \Rightarrow \Psi = \frac{\Gamma}{2\pi}\ln\rho + C = \frac{\Gamma}{2\pi}\ln\left(\frac{\rho}{\rho_0}\right)$$

for a particular streamline, $\Psi = \text{constant} = \Psi_0$

$$\ln\frac{\rho}{\rho_0} = \frac{\Psi_0 2\pi}{\Gamma} \Rightarrow \rho = \rho_0 e^{\Psi_0 2\pi/\Gamma} = \text{constant}$$



c) Determine whether this flow conserves mass.

$$\vec{\nabla}\cdot\vec{u} = \frac{1}{\rho}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{\rho}\frac{\partial \rho u_\rho}{\partial\rho} = 0 \quad \therefore \text{ this flow conserves mass.} \qquad \nabla^2\phi = \nabla^2\left(\frac{-\Gamma}{2\pi}\theta\right) = 0$$

d) Determine whether this flow has vorticity in any location other the origin

$$\vec{\nabla}\times\vec{u} = -\hat{\rho}\cancel{\frac{\partial u_\theta}{\partial z}}^{0} + \hat{z}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\left(\frac{-\Gamma}{2\pi\rho}\right) = \hat{z}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{-\Gamma}{2\pi}\right) = 0 \quad \therefore \text{ this flow doesn't have vorticity.}$$

How can you justify the result of question (b) with the result of this question?

I googled it and found a wiki page describing free (irrotational) vortex and forced (rotational) vortex. The angular ~~momentum~~ velocity, ρu, is uniform everywhere throughout the flow; except at $\rho = 0$.

e) It turns out that stream line Bernoulli between any two points in an irrotational flow, not just along a streamline.

Does the streamline Bernoulli eq. & normal Bernoulli equation result in the same prediction of pressure at different locations in this flow?

Yes

Why or why not?

Normal forces to streamline

$$a_n = \frac{u^2}{\rho} \qquad \rho_{fluid}(a_n) = \rho_{fluid}\left(\frac{u^2}{\rho}\right) = \frac{-\partial}{\partial n}\left(P + \rho_{fluid}\, g z\right)$$

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial \rho} = \frac{-\Gamma}{2\pi \rho^2} = \frac{u}{\rho}$$

$$-\frac{\partial}{\partial n}\left(P + \rho_{fluid}\, g z\right) - \rho_{fluid}\frac{u^2}{\rho} = 0$$

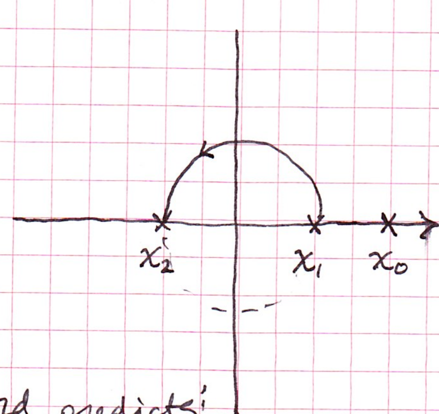
$$-\frac{\partial}{\partial n}\left(P + \rho_{fluid}\, g z + \rho_{fluid}\frac{u^2}{2}\right) = 0 \therefore P + \rho_{fluid}\, g z + \rho_{fluid}\frac{u^2}{2} = \text{constant}$$

Show using an example:

everywhere on x-axis, $\theta = 0 \therefore \Phi = 0 \quad \vec{u}(x,0) = \frac{-\Gamma}{2\pi x}\hat{y}$

$u^2 = \frac{\Gamma^2}{4\pi^2 x^2}$ let $z=0$, 2D flow define $g \perp$ to x,y $\therefore \rho g z = 0$

$$P_{x_1} + \left(\frac{\Gamma}{2\pi}\right)^2 \frac{\rho_{fluid}}{2}\frac{1}{x_1^2} = P_{x_2} + \left(\frac{\Gamma}{2\pi}\right)^2 \frac{\rho_{fluid}}{2}\frac{1}{x_2^2}$$



$x_1 = x_0/2$
$x_2 = -x_1$

normal coord predicts:

$$P_{x_0} + \left(\frac{\Gamma}{2\pi}\right)^2 \frac{\rho_f}{2}\frac{1}{x_0^2} = \underbrace{P_{x_1} + \left(\frac{\Gamma}{2\pi}\right)^2 \frac{\rho_f}{2}\frac{1}{x_1^2} = P_{x_2} + \left(\frac{\Gamma}{2\pi}\right)^2 \frac{\rho_f}{2}\frac{1}{x_2^2}}_{\text{streamline predicts.}} \cancel{= P_{x_0} + \left(\frac{\Gamma}{2\pi}\right)^2 \frac{\rho_f}{2}\frac{1}{x_0^2}}$$