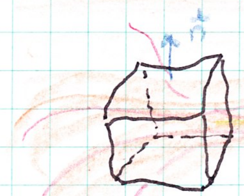
Meg A. Noah

1. Concept Questions

a) Describe how (conceptually) $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$ is the conservation of mass statement for a compressible fluid.

Consider a fluid with a variable mass density $\rho(x,y,z,t)$ and a variable current density $\rho\vec{u}(x,y,z)$. Let V be an arbitrary volume within the fluid bounded by a piecewise smooth closed surface S. Considering the total amount of mass inside V and the net amount ~~entering V (or exiting)~~ per unit time through surface S. Conservation of mass says:

$$\frac{\partial}{\partial t}\iiint_V \rho(x',y',z',t)\,d\tau' = -\oint_S \rho\vec{u}(x',y',z')\cdot d\vec{S}$$



V

By Gauss' Law:

$$\frac{\partial}{\partial t}\iiint_V \rho(x',y',z',t)\,d\tau' = -\iiint_V \vec{\nabla}\cdot(\rho\vec{u})\,d\tau'$$

$$\iiint_V \left\{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u})\right\} d\tau' = 0$$

Since the volume V can be arbitrarily big or small, the integrand has to be zero.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

$\therefore$ this expression results when conservation of mass is assumed. (whether compressible or incompressible)

Reference: My own math notes, deriving the continuity equation from conservation of charge!

b) Describe how (conceptually) $\nabla\cdot\vec{u} = 0$ is the conservation of mass for an incompressible fluid.

Start with $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$ if ρ = constant (incompressible) then $\vec{\nabla}\cdot\vec{u} = 0$.

c) For the convective/advective acceleration term answer the following questions:

i) Write out the full expression for: $\vec{u}\cdot\vec{\nabla}\vec{u}$

$$(\vec{u}\cdot\vec{\nabla})\vec{u} = \left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}\right)\vec{u} = \left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}\right)(u_x\hat{x} + u_y\hat{y} + u_z\hat{z})$$

$$= \hat{x}\left\{u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right\} + \hat{y}\left\{u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z}\right)$$

$$+ \hat{z}\left\{u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z}\right\}$$

c) ii) Demonstrate whether or not the following statement is true, $\vec{u}\cdot\vec{\nabla}\vec{u} = \vec{u}\,\vec{\nabla}\cdot\vec{u}$

$$\vec{u}(\vec{\nabla}\cdot\vec{u}) = (u_x\hat{x} + u_y\hat{y} + u_z\hat{z})\left[\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(u_x\hat{x} + u_y\hat{y} + u_z\hat{z}\right)\right]$$

$$= u_x\hat{x} + u_y\hat{y} + u_z\hat{z}\left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right]$$

$$= \hat{x}\left\{u_x\frac{\partial u_x}{\partial x} + u_x\frac{\partial u_y}{\partial y} + u_x\frac{\partial u_z}{\partial z}\right\} + \hat{y}\left\{u_y\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_z}{\partial z}\right\}$$

$$+ \hat{z}\left\{u_z\frac{\partial u_x}{\partial x} + u_z\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_z}{\partial z}\right\}$$

generally $\vec{u}\cdot\vec{\nabla}\vec{u} \neq \vec{u}\,\vec{\nabla}\cdot\vec{u}$

example: $\vec{u} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\vec{u}\cdot\vec{\nabla}\vec{u} = x\hat{x} + y\hat{y} + z\hat{z} \qquad \vec{u}(\vec{\nabla}\cdot\vec{u}) = 3x\hat{x} + 3y\hat{y} + 3z\hat{z}$$

d) Find and describe a unique example where Bernoulli in streamline-normal coordinates is a good method to describe the flow.

(I made some movies of egg separation thru glass funnel & used dye to show it forms streams).

Separation of an egg yolk from an egg white without introducing yolk into the white is necessary for many recipes. Different devices using pressure or, like in the 'Snot a mug', gravity to force the white out of the mixture, have been invented. Yolk and white have similar densities, the yolk is protected by a membrane but that can break if the flow rate is too rapid. S-N coordinates describe the flow of the white around the yolk because it is laminar, and Bernoulli streamlines

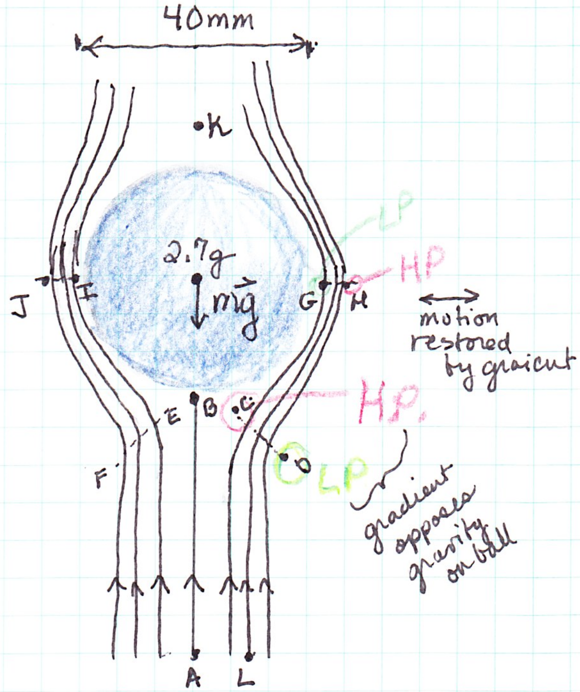
Home > Birchstone Studios > 'Snot a Mugs

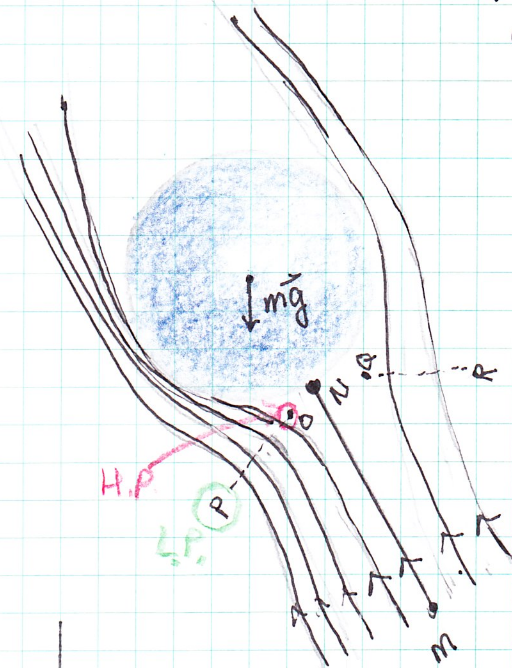
You either love this stuff or you hate it, but you still can't look away... Birchstone Studios has the most *eggstraordinary* pottery (wickedgoodcandles.com)

This handcrafted pottery for your kitchen is beautiful, practical, and just a little disgusting. Crack an egg into his head, tip him forward a bit, and he separates the egg whites through his nostrils. A very unusual item... I bet you don't have one, yet.

show why the neutrally buoyant yolk

e) A well-known 'trick' to demonstrate pressure and the Bernoulli equation is the levitation of a Ping-Pong Ball in a stream of air generated by a hair dryer. Describe.





* Note: This can also be explained with C.V. and cons. of momentum but question specifically called for Bernoulli stream.

velocity along streamline:

$V_A = V_0$ $\quad V_B = 0$ (stagnation point)

$V_G > V_0$ $\quad V_I > V_0$

Kinetic Pressure along A → B $\frac{1}{2}\rho V_0^2$

along streamline L

Gravity term (negligable because it is air)

$\rho g z_K > \rho g z_B > \rho g z_A$ linear...

When blown at an angle to the downward force of gravity, the ball can still be baloneed even though the stagnation point at N is not directly below the center of mass of the ball.

The streamlines that pass underneath the ball create a large pressure gradient Hi → Lo from O to P. This opposes the gravitational & drag forces of the ball keeping it levitated (and spinning).

Pressure P_B is the highest pressure

The normal line from C → D is Hi → Lo

The normal line from G → H is Lo → Hi

The stagnation point B has the highest pressure on the order of $\frac{1}{2}\rho V_0^2$ with hair dryer speed of ~10 m/s, it is on the order of 50 Pa (gage). The pressure difference between B and K and the gradient E → F and C → D gives the ball lift. It balances with the net downward force of the 2.7g ball and drag opposing upward ~~flow~~ motion. Gravity of ball over 40mm diameter is also on the order of 20-25 N/m²

The motion of the ball ~~betwe~~ ~~per~~ perpendicular to the airflow is evident. The LP → HP gradient from G → H and I → J provide a restoring force to that oscillation.

The ball spins on an axis perpendicular to $\vec{V}$.

Reference: Class Notes.

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f) manometers

(f) In the two diagrams below, draw and shade the level of mercury in the manometers (as accurately as possible). Assume that viscous effects are negligible:

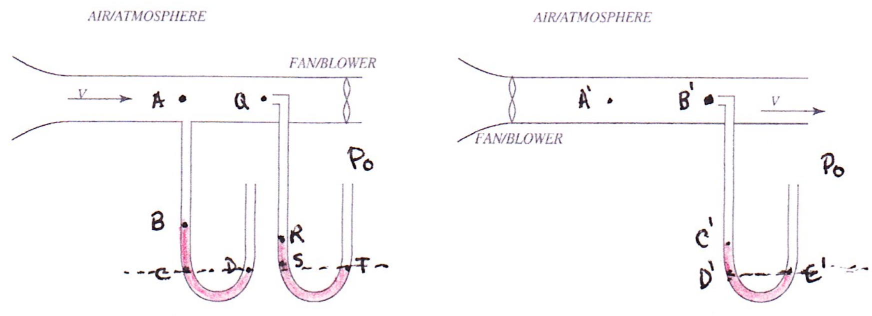

Figure 2: (Left) A fan downstream of the measurement section (Right) a fan upstream of the measurement section

Left

	Point	ρ	Velocity	Pressure
	A	ρ_A	V	$P_A < P_o$
	B	ρ_A/ρ_{Hg}	0	$P_B = P_A + \rho_A g(z_A - z_B)$
	C	ρ_A/ρ_{Hg}	0	$P_C = P_B + \rho_{Hg} g(z_B - z_C) = P_A + \rho_A g(z_A - z_B) + \rho_{Hg} g(z_B - z_C)$
	D	ρ_{Hg}/ρ_A	0	$P_D = P_o = P_C$
stagnation pt.	Q	ρ_A	0	$P_Q = P_A + \frac{1}{2}\rho_A V^2$ (Bernoulli: dynamic pressure)
	R	ρ_A/ρ_{Hg}	0	$P_R = P_Q + \rho_A g(z_A - z_R)$ $[z_Q = z_A]$
	S	ρ_{Hg}	0	$P_S = P_R + \rho_{Hg} g(z_R - z_S) = P_A + \frac{1}{2}\rho_A V^2 + \rho_A g(z_A - z_R) + \rho_{Hg} g(z_R - z_S)$
	T	ρ_{Hg}/ρ_A	0	$P_T = P_S = P_o$

static pressure

$$z_B - z_C = [P_o - P_A - \rho_A g(z_A - z_B)]/\rho_{Hg} g$$

$$z_R - z_S = [P_o - P_A - \tfrac{1}{2}\rho_A V^2 - \rho_A g(z_A - z_R)]/\rho_{Hg} g$$

$$(z_B - z_C) > (z_R - z_S)$$

Right

	Point	ρ	Velocity	Pressure
	A'	ρ_A	V_-	$P_{A'} < P_o$
stagnation pt.	B'	ρ_A	0	$P_{B'} = P_{A'} + \frac{1}{2}\rho_A V_-^2$ (Bernoulli)
	C'	ρ_A/ρ_{Hg}	0	$P_{C'} = P_{B'} + \rho_A g(z_{A'} - z_{C'})$ $[z_{B'} = z_{A'}]$
	D'	ρ_{Hg}	0	$P_{D'} = P_{C'} + \rho_{Hg} g(z_{C'} - z_{D'}) = P_{A'} + \frac{1}{2}\rho_A V_-^2 + \rho_A g(z_{A'} - z_{C'}) + \rho_{Hg} g(z_{C'} - z_{D'})$
	E'	ρ_{Hg}/ρ_A	0	$P_{D'} = P_{E'} = P_o$

$$z_{C'} - z_{D'} = [P_o - P_A - \tfrac{1}{2}\rho_A V_-^2 - \rho_A g(z_A - z_{C'})]/\rho_{Hg} g$$

note: the velocity at A' (right) is slightly less than the velocity at A (left) because sucking is more efficient than blowing at organizing the flow. $\therefore (z_{C'} - z_{D'}) > (z_R - z_S)$