$$\vec{J} = \vec{L} + \vec{S} \; ? \qquad m_\ell + m_s = m \qquad \hbar m = \hat{J}_z$$

$$\boxed{J_z = L_z + S_z \Rightarrow \text{these don't commute} \quad [J_x, J_y] = i\hbar J_z \qquad \hbar m_\ell = \hat{L}_z \quad \hbar m_s = \hat{S}_z}$$

Projection Theorem $\quad \langle \alpha j' m' | A_q | \alpha j m \rangle = \cancel{\langle \alpha j' m' | \vec{J} \cdot \vec{A} | \alpha j m \rangle}$

$$= \frac{\langle \alpha j' m' | \vec{J} \cdot \vec{A} | \alpha j m \rangle \langle \alpha j' m' | J_q | \alpha j m \rangle}{\hbar^2 j(j+1)}$$

Use here $\vec{A}$ is $\vec{S}$

$$\langle n \ell \tfrac{1}{2} j m | S_z | n \ell \tfrac{1}{2} j m \rangle = \frac{\langle n \ell \tfrac{1}{2} j m | \vec{J} \cdot \vec{S} | n \ell \tfrac{1}{2} j m \rangle \overbrace{\langle n \ell m \tfrac{1}{2} j m | J_z | n \ell \tfrac{1}{2} j m \rangle}^{\hbar m}}{\hbar^2 j(j+1)}$$

$\vec{J} = \vec{L} + \vec{S}$

$\vec{L} = \vec{J} - \vec{S} \qquad L^2 = (\vec{J} - \vec{S})^2 = J^2 + S^2 - 2\vec{J} \cdot \vec{S} \Rightarrow$

$$\vec{J} \cdot \vec{S} = \tfrac{1}{2}(-L^2 + J^2 + S^2)$$

$$\tfrac{1}{2} \langle n \ell \tfrac{1}{2} j m | J^2 + S^2 - L^2 | n \ell \tfrac{1}{2} j m \rangle = \tfrac{1}{2}\Big[j(j+1) + \underbrace{\tfrac{1}{2}(\tfrac{1}{2}+1)}^{3/4} - \ell(\ell+1) \Big] \hbar^2$$

$$\langle S_z \rangle = \frac{\cancel{\hbar^2}(j(j+1) + 3/4 - \ell(\ell+1))\hbar m}{2\cancel{\hbar^2} j(j+1)} = \frac{\hbar m}{2} \frac{[j(j+1) + 3/4 - \ell(\ell+1)]}{j(j+1)}$$

$$\therefore \; E^{(1)}_{n\ell\frac{1}{2}jm} = \frac{eB\hbar m}{2\mu c} + \underbrace{\frac{eB}{2\mu c}}_{\text{Bohr magnetron} \times \text{field} = \text{energy}} \frac{[j(j+1) + 3/4 - \ell(\ell+1)]\hbar m}{2j(j+1)}$$

let $j = \ell \pm \tfrac{1}{2}$

$$E^{(1)}_{n\ell\frac{1}{2}jm} = \frac{eB\hbar m}{2\mu c}\left[1 + \frac{j(j+1) + 3/4 - \ell(\ell+1)}{2j(j+1)}\right]$$

$$= \frac{eB\hbar m}{2\mu c}\left[1 \pm \frac{1}{2\ell+1}\right] \text{ for } j = \ell \pm \tfrac{1}{2}$$

(when $j = \ell + \tfrac{1}{2}$: $\frac{eB}{\mu c}\hbar m$ Bigger splitting)

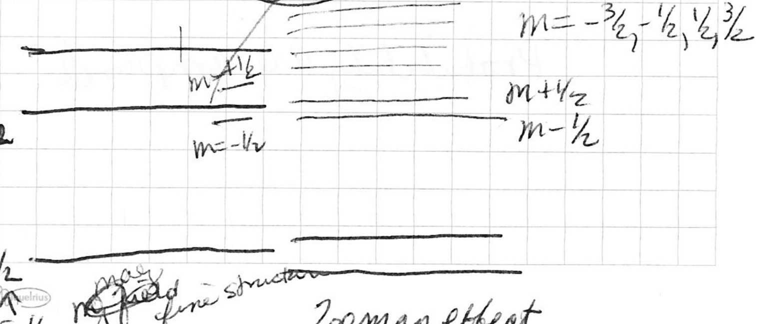
(when $j = \ell - \tfrac{1}{2}$)

What does this predict

$$2P_{1/2} = \frac{eB\hbar m}{2\mu c}\left[1 - \tfrac{1}{3}\right] = \frac{2}{3}\frac{eB\hbar m}{2\mu c}$$

fine structure $2P_{3/2}$

$2P_{1/2}$, $2S_{1/2}$



NOTE BOOK

Time Dependent Perturbation Theory

2nd exam is next Thursday April 16th — wants to include in 2nd exam.

\#1 18.2.1 - 18.2.5 18.2.6 — Collect on ~~Tuesday~~ Thursday

\#2 18.4.1 - 18.4.4, 18.5.1, 18.5.2

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H|\psi(t)\rangle$$

$$|\psi(t)\rangle = |\psi(0)\rangle e^{-iE_n t/\hbar}$$

$$E_n|\psi_n(0)\rangle = H|\psi_n(0)\rangle \qquad \text{T.I.S.E.}$$

$$|\psi_n(0)\rangle = |\phi_n\rangle$$

$$|\psi(0)\rangle = \sum_n c_n|\phi_n\rangle \qquad |\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar}|\phi_n\rangle$$

$$H = H^0 + H^t = H^0 + V(t)$$

assume we can solve H^0 exactly H^0 has no time dependence.

$V(t)$ could be interaction with electromagnetic wave

at $t=0$ system prepared in eigenstate H^0 say $|i\rangle_0$

$$H^0|i\rangle_0 = E_i^0|i\rangle_0$$

~~Beca~~ If $V(t)$ was absent - it will remain in that state forever. (just a phase factor $e^{-iEt/\hbar}$).

What is the probability of finding it in another eigenstate of H^0?
$|f\rangle_0$ at time t

$$H_0|f\rangle_0 = E_f^{(0)}|f\rangle_0$$

Probability are amplitud

H^0 eigenstates form a complete set ∴

State vector at time t can be rep as expansion of

$$|\Psi(t)\rangle = \sum_n c_n(t) |n\rangle_0$$

$$H = H^0 + V(t)$$

If $V(t) = 0 \quad C_n(t) = c_n(0) e^{-iE_n t/\hbar}$

Could write something like

$$C_n(t) = d_n(t)\, e^{-iE_n^0 t/\hbar}$$

Substitute above $|\Psi(t)\rangle = \sum_i d_n(t) e^{-iE_n^0 t/\hbar} |n\rangle_0$

get two terms when substitute in TDSE

LHS $\quad i\hbar \sum_n \left(\frac{dd_n}{dt}\right) e^{-iE_n^{(0)}t/\hbar} |n\rangle_0 + i\hbar \sum_i d_n(t) \left(\frac{-iE_n^{(0)}}{\hbar}\right) e^{-iE_n^{(0)}t/\hbar} |n\rangle_0$

RHS

$$H^0 |\Psi(t)\rangle = \sum_i d_n(t) e^{-iE_n t/\hbar} E_n^0 |n\rangle_0$$

$+$

$$V|\Psi(t)\rangle = \sum d_n(t) e^{-iE_n t/\hbar} V(t)|n\rangle_0$$

only 1 term LHS & one RHS survive

$$i\hbar \sum \dot{d}_n e^{-iE_n^0 t/\hbar} |n\rangle_0 = \sum d_n e^{-iE_n t/\hbar} V(t)|n\rangle_0$$

take scalar product of both sides w/ bra vector ~~…~~ ${}_0\langle f|$

${}_0\langle f|n\rangle_0 = \delta_{fn}$

~~$i\hbar \dot{d}_f e^{-}$~~ ~~$i\hbar \dot{d}$~~ $i\hbar \dot{d}_f e^{-iE_f t/\hbar} = \sum_n d_n e^{-iE_n t/\hbar} {}_0\langle f|V(t)|n\rangle_0$

$$\dot{d}_f = \frac{1}{i\hbar} \sum_n d_n e^{i\omega_{fn} t} {}_0\langle f|V(t)|n\rangle_0$$

exact relation, no approx as a good as TDSE

$\omega_{fn} = \dfrac{E_f^0 - E_n^0}{\hbar}$

↑ infinite # of terms

if V is absent $\dot{d}_f = 0$ and it is time indep.

at $t=0$ $|i\rangle_0 = \psi$

$d_n(0) = \delta_{ni}$ because it was definitely in that state

$\dot{d}_f \simeq \delta_{ni}$ ~~$\dot{d}_f \simeq \delta_{ni} = \frac{1}{i\hbar}$~~

$$\dot{d}_f \simeq \frac{1}{i\hbar}\sum_n \delta_{ni}\, e^{i\omega_{fn}t}\, {}_0\langle f|V(t)|n\rangle_0 = \frac{1}{i\hbar} e^{i\omega_{fi}t}\, {}_0\langle f|V(t)|i\rangle_0$$

$$d_f(t) = d_f(t_0) + \frac{1}{i\hbar}\int_{t_0}^{t} e^{i\omega_{fi}t}\, {}_0\langle f|V(t')|i\rangle_0\, dt'$$

this is the 1ST order approximation

at $t=0$ only one state is occupied

can improve go back to exact eq.

$$d_f(t) = \delta_{fi} + \frac{1}{i\hbar}\int_{t_0}^{t} e^{-i\omega_{fi}t}\, {}_0\langle f|V(t)|i\rangle_0\, dt'$$

if $f \neq i$

$d_f(t) =$ this is the probability amplitude

$$C_f(t) = d_f(t)\, e^{-iE_f^0 t/\hbar}$$

$$P_f(t, t_0) = |C_f(t)|^2 = |d_f(t)|^2$$

probably can do all problems in 1ST set

1-D SHO

$V(t) = -eEX\, e^{-t^2/\tau^2}$

$H = H_0 + V(t)$

$|i\rangle_0$ at time $t = t_0$

$$d_f(t,t_0) = \delta_{fi} - \frac{i}{\hbar}\int_{t_0}^{t} {}_0\langle f|V(t')|i\rangle_0 \, e^{i\omega_{fi}t'} dt'$$

$$\omega_{fi} = \frac{E_f^{(0)} - E_i^{(0)}}{\hbar}$$

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

$V = -eEx\,e^{-t^2/\tau^2}$ perturbation like that apply uniform E-field

potential energy

Suppose particle is SHO in ground state

What is prob of finding it in excited state? $E_n^{(0)} = (n+\tfrac{1}{2})\hbar\omega$ $n = 0,1,2,\ldots$

$$d_n = \delta_{n0} - \frac{i}{\hbar}\int_{-\infty}^{+\infty} {}_0\langle n| -eEx\,e^{-t^2/\tau^2}|0\rangle_0 \, e^{i\omega_{fi}t'} dt'$$

$d_n(-\infty)$ $e^{i\omega_{fi}t'} \Rightarrow e^{in\omega t'}$

$$= \delta_{n0} + \frac{i}{\hbar}eE\int_{-\infty}^{+\infty} {}_0\langle n|x|0\rangle_0 \, e^{in\omega t' - t'^2/\tau^2} dt$$

x can only connect if the n are diff by one unit $\quad x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$

$a^\dagger|0\rangle = 1$ only when $n=1$ is it non-zero $\quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle = |1\rangle$

$${}_0\langle 1|x|0\rangle_0 = \sqrt{\frac{\hbar}{2m\omega}}\,{}_0\langle 1|1\rangle = \sqrt{\frac{\hbar}{2m\omega}} \quad n=1$$

$$d_1(\infty) = +\frac{ieE}{\hbar}\sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{+\infty} e^{in\omega t'} e^{-t'^2/\tau^2} dt' \qquad \int_{-\infty}^{+\infty} e^{-(t'^2/\tau^2 - i\omega t')} dt'$$

$\int_{-\infty}^{+\infty} \cancel{e^{-(\ldots)}}$

Complete the squares

$$\frac{t'^2}{\tau^2} - i\omega t' = \left(\frac{t'}{\tau} + a\right)^2 - a^2 = \frac{t'^2}{\tau^2} + 2a\frac{t'}{\tau}$$

$$\frac{2a}{\tau} = -i\omega \qquad a = \frac{-i\omega\tau}{2}$$

$$\frac{t'^2}{\tau^2} - i\omega t' = \left(\frac{t'}{\tau} - \frac{i\omega\tau}{2}\right)^2 + \frac{\omega^2\tau^2}{4}$$

$$d_1(\infty) = \frac{ieE}{\hbar}\sqrt{\frac{\hbar}{2m\omega}}\, e^{-\omega^2\tau^2/4}\int_{-\infty}^{+\infty} e^{-(t'/\tau - i\omega\tau/2)^2} dt'$$

make a change of variables

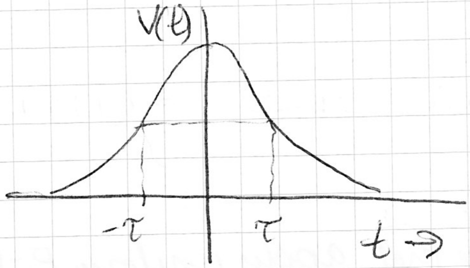
$$y = \frac{t'}{\tau} - \frac{i\omega\tau}{2} \qquad dy = \frac{dt'}{\tau}$$

$$\int_{-\infty}^{+\infty} \tau e^{-y^2} dy = \tau\sqrt{\pi}$$

$$d_1(\infty) = \frac{ieE}{\hbar}\sqrt{\frac{\hbar}{2m\omega}}\, e^{-\omega^2\tau^2/4}\,\tau\sqrt{\pi} \qquad P_{1(\infty)} = \frac{e^2E^2\, e^{-\omega^2\tau^2/2}}{\hbar(2m\omega)}\tau^2\pi$$

various things we can learn from here

$V(t) = -eEx\,e^{-t^2/\tau^2}$ electric field has gaussian time dependence



when $t=\tau$

like the width of

when $\tau \to 0$ sharp peak (sharply peaked) → probability goes to zero

$\tau \to \infty$ flattens out

adding $\tau \to 0$ finite perturbation at $t=0$

$\tau \to \infty$ then $V(t) \to eEx$ perturbation is independent of x

only the wave ftn v energies are shifted by constant

Sudden Approximation don't need perturbation theory

Hamiltonian changes abruptly

$$i\hbar \frac{\partial}{\partial t}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle$$

$$i\hbar \int_{-\varepsilon}^{+\varepsilon} \frac{d}{dt}|\Psi(t)\rangle dt = \int_{-\varepsilon}^{+\varepsilon} H(t)|\Psi(t)\rangle dt$$

RHS $\to 0$

$$LHS \to i\hbar\left[\Psi(t{=}{+}\varepsilon) - \Psi(t{=}{-}\varepsilon)\right]$$

$$\therefore \Psi(t=+\varepsilon) = \Psi(t=-\varepsilon)$$

Very, very short time

$H^3 \to He^{3+}$

assume Ψ doesn't change

ground state of He^{3+} might have done that

β decay e^- flies out highly relativistic

$m_n c^2 = 141$ MeV

$m_p c^2 = 137$ MeV

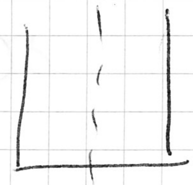
$\Delta = 4$ MeV

$.511$ MeV $= m_e c^2$

$\therefore e^-$ is highly relativistic

Book calculates the ratio
Characteristic time
$t = Z\alpha \qquad \alpha = \left(\frac{1}{137}\right)$
for low Z atoms this works

Adiabatic Perturbation - exact opposite - happens very slowly
change happens very slowly
example: 1-D infinite potential well



move walls very slowly
particle in ground state of the new well
the time to make change >> characteristic time $\quad -a/2 \qquad +a/2$

time to make one round trip

$E_n = \frac{\hbar^2\pi^2 n^2}{2ma^2} \quad n = 1, 2, 3, \ldots$ look at

momentum is $\frac{\hbar\pi}{a} = p_0 \; n=1 \qquad \tau = \frac{a}{v} = \frac{am}{p} = \frac{ma^2}{\hbar\pi}$

happens in time much larger than τ

$\left(\frac{dL}{dt}\right)\tau \qquad \frac{\left(\frac{da}{dt}\right)\tau}{a} \ll 1 \qquad \frac{da}{dt}\cdot\frac{ma^2}{\hbar\pi}\frac{1}{a} = \frac{da}{dt}\left(\frac{ma}{\hbar\pi}\right) \ll 1$

$\frac{\hbar\pi}{ma}$ is the average velocity $\quad \frac{da}{dt}$ = velocity of the walls $\quad \frac{V_{wall}}{\langle V_p\rangle} \ll 1$ condition for adiabatic change

wave function changes in a very prescribed way.

$V(t) = V_0 e^{\pm i\omega t}$ there are many cases where this will happen
V_0 is a function of position/momentum variables of e^-
energy increases) $\quad e^{+i\omega t} \quad e^{-i\omega t}$ transition to lower energy state

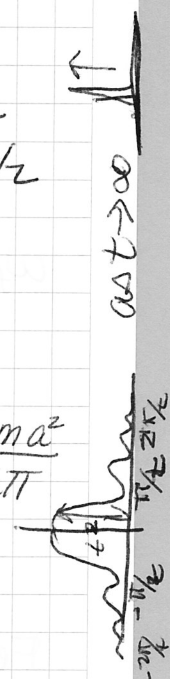
example: $V = V_0 e^{-i\omega t} \quad V_0 = V_0(\vec{x}, \vec{p}, \text{etc.}) \quad H = H_0 + V(t)$
at $t=0$ system was in state $|i\rangle$ an eigenstate of H_0

$H_0|i\rangle_0 = E_i^{(0)}|i\rangle_0$ whats the probability of finding it in f at time t

$d_f(t) = \delta_{fi} - \frac{i}{\hbar}\int_0^t \langle f|V_0(\vec{x},\vec{p})|i\rangle e^{-i\omega t'} e^{+i\omega_{fi}t'} dt' \overset{f\neq i}{=} \cancel{\delta_{fi}} - \frac{i}{\hbar}\langle f|V_0|i\rangle \int_0^t e^{+i(\omega_{fi}-\omega)t'} dt'$

$d_f(t) = -\frac{i}{\hbar}\langle f|V_0|i\rangle \frac{\left(e^{i(\omega_{fi}-\omega)t} - 1\right)}{i(\omega_{fi}-\omega)} \qquad P(f_0 \text{ at } t) = \frac{|\langle f|V_0|i\rangle|^2}{\hbar^2(\omega_f-\omega)^2}\left[2 + e^{i(\omega_f-\omega)t} - e^{-i(\omega-\omega)t}\right]$

$P(f \text{ at } t) = \frac{|\langle f|V|i\rangle|^2\, 4\sin^2\left(\frac{(\omega_{fi}-\omega)}{2}t\right)}{\hbar^2\left(\frac{\omega_{fi}-\omega}{2}\right)^2}$



as $t \to \infty$ $\quad$ area $= t^2\left(\frac{2\pi}{t}\right) = \pi t$

as $t \to \infty$ dirac delta $\frac{\sin^2 xt}{x^2} = \pi t \delta(x)$ definition even

$\left(\frac{1}{10^{-18}}\right) s$ infinite amount of time to an atom.

$$P_f(t) = \frac{1}{\hbar^2} |\langle f|V_0|i\rangle|^2 \pi t \delta(x) \quad \text{where } x = \left(\frac{\omega_{fi} - \omega}{2}\right)$$

$$\delta(ax) = \frac{1}{|a|} \delta(x)$$

$$= \frac{1}{\hbar^2} |\langle f|V_0|i\rangle|^2 2\pi t \delta(\omega_{fi} - \omega)$$

$$\omega_{fi} = \frac{(E_{f0} - E_{i0})}{\hbar} \qquad \delta(\omega_{fi} - \omega) = \delta\left(\frac{E_{f0} - E_{i0} - \hbar\omega}{\hbar}\right)$$

$$P_f(t) = \frac{\hbar}{\hbar^2} |\langle f|V_0|i\rangle|^2 \frac{2\pi t}{} \delta(E_f^{(0)} - E_i^{(0)} - \hbar\omega)$$

atom can only absorb in units of $\hbar\omega$

Fermi's Golden Rule $W_f(t) \equiv$

Rate of Transition Probability

conservation of energy absorbed

$$W_f(t) = \frac{dP_f(t)}{dt} = \frac{2\pi}{\hbar} |{}_0\langle f|V_0|i\rangle_0|^2 \delta(E_f^{(0)} - E_i^{(0)} - \hbar\omega)$$

used $e^{-i\omega t} \to \underline{e^{+i\omega t}}$ emission $\delta(E_f^{(0)} - E_i^{(0)} + \hbar\omega)$

absorption

$E_f^{(0)} = E_i^{(0)} - \hbar\omega$

in both cases $\delta(E_f^{(0)} - E_i^{(0)} \pm \hbar\omega)$

two important things 1) energy is conserved

2) independent of time

No experiment can measure frequency exactly

April 21

Higher orders easier to calculate

$T_I(t,t_0) =$

you know state at time t_0 $\quad |\Psi(t)\rangle_I = T_I(t,t_0)|\Psi(t_0)\rangle_I$

can always obtain S picture from I picture

at t_0 $|\Psi(t_0)\rangle_I = |\Psi(t_0)\rangle_S$

$$T_I(t,t_0) = \mathbb{1} + \frac{1}{i\hbar}\int V_I(t_1')\,dt_1' + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^{t} dt_1\, V_I(t_1)\int_{t_0}^{t_1} V_I(t_2)\,dt_2 + \ldots$$

3RD order has triple integral

$$V_I = e^{iH_0t/\hbar}\, V_S\, e^{-iH_0t/\hbar}$$

once we know the time evolution operator

$|i\rangle_0$

what's the prob of finding it in $|f\rangle_0$ at time t

$T_I(t,t_0)|i\rangle_0$

$T_f(t,t_0) = {}_0\langle f|T_I(t,t_0)|i\rangle_0$ $\qquad$ can call it d_f ... as before

$$P_f(t) = |T_f(t,t_0)|^2 = |{}_0\langle f|T_I(t,t_0)|i\rangle_0|^2$$

if V has a harmonic time dependence, it turns out that for large time $t \gg 1/\omega$ the transition probability is proportional to time

Fermi's Golden Rule

$$w_f = \frac{2\pi}{\hbar}|{}_0\langle f|V_0|i\rangle_0|^2\,\delta(E_f^{(0)} - E_i^{(0)} \pm \hbar\omega)$$

$V(t) = V_0 e^{\pm i\omega t}$ $\qquad$ if "+" $E_f^0 = E_i^0 - \hbar\omega$ emission

$t \gg 1/\omega$ $\qquad$ "−" $E_f^0 = E_i^0 + \hbar\omega$ absorption

if the energy is the same, $\omega = 0$,
Scattering final energy = initial energy elastic
but state is different $|\vec{p}| =$ same but dif direction

if $\omega = 0$ incident monochromatic electromagnetic radiation

rate of stimulated emission & absorption is same $|\langle f|V_0|i\rangle|^2$

consider

Maxwell's Equations Gaussian Units! in vacuum

Inhomogeneous $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$ homogeneous vector potential

Amp $\vec{\nabla}\times\vec{B} - \underbrace{\frac{1}{c}\frac{\partial\vec{E}}{\partial t}}_{\text{added by Maxwell, displacement current}} = \frac{4\pi}{c}\underbrace{\vec{J}}_{\text{current density}}$

$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$ faraday

$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = 0$

$\vec{\nabla}\times\left(E + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = 0$

$\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi \Rightarrow \vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$

once you chose $\vec{B} = \vec{\nabla}\times\vec{A}$

get equations for potentials

$\underbrace{\vec{\nabla}\cdot(-\nabla\phi}_{\text{Laplacian}} - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}) = 4\pi\rho$

$-\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = 4\pi\rho$

$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) - \frac{1}{c}\frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = \frac{4\pi}{c}\vec{J}$

$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} + \frac{1}{c}\vec{\nabla}\frac{\partial\phi}{\partial t} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{J}$

freedom in chosing the potentials

Force is given by Lorenz Force equation $\vec{F} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}$

freedom in chosing $\vec{A}$ and ϕ

$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\chi$

$\vec{\nabla}\times A' = \vec{\nabla}\times\vec{A}$

$\vec{A}' = \vec{A} + \vec{\nabla}\chi$

the new electric field...

$\cancel{\phi \to \phi' = \phi - \frac{1}{c}\frac{\partial}{\partial t}(\vec{A}')}$ $\qquad \phi' = \phi + \frac{1}{c}\frac{\partial \chi}{\partial t}$ Gauge transformation

can show $\vec{E}' = \vec{E}$ by this transformation

$$\vec{E}' = -\nabla\phi' - \frac{1}{c}\frac{\partial}{\partial t}(\vec{A} + \vec{\nabla}\chi) = \vec{E}$$

Two Gauges:

Coulomb Gauge $\quad \vec{\nabla}\cdot\vec{A} = 0 \quad$ can always add a gradient

uncouples $\nabla^2\phi = -4\pi\rho$ so it is Poisson's equation.

time + space dependence

and $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A} = \frac{4\pi}{c}\vec{J} - \frac{1}{c}\vec{\nabla}\frac{\partial\phi}{\partial t}$

suppose we have a region of space $\rho = 0, \vec{J} = 0$

$\phi \to 0$ at $r \to \infty$ then ϕ is zero everywhere

simple to solve

$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \nabla^2\vec{A} = 0$ wave equation

every component of $\vec{A}$ has variable wavelength solutions.

$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla}\times\vec{A}$

so in space where no charge / no current can have e/m waves

~~Both~~ Poynting Vector $\quad$ energy per unit area per unit time

$\vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B})$ $\qquad \vec{E}$ and $\vec{B}$ have the same units in gaussian

vector because energy flow has a direction

$\vec{H} = \frac{\vec{B}}{\mu_0}$ don't need $\vec{H}$ or $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

$\vec{D} = \epsilon\vec{E}$

if $\vec{P} = \epsilon_0\chi\vec{E}$ then

$\vec{D} = \cancel{\epsilon_0 \vec{E}}\ \cancel{(1+\epsilon_0)}\ \epsilon_0(1+\chi)\vec{E}$

$(1+\chi) = \cancel{K}$

dielectric constant

trial solution:

$\vec{A} = \vec{A}_0\cos(\vec{k}\cdot\vec{x} - \omega t)$

can see that

$\frac{\partial^2\vec{A}}{\partial t^2} = -\omega^2\vec{A} \qquad \nabla^2 A$

$$\frac{\omega^2}{c^2}\vec{A}_0\cos(\vec{k}\cdot\vec{x} - \omega t) - k^2\vec{A}_0\cos(\vec{k}\cdot\vec{x} - \omega t) = 0$$

$\frac{\omega^2}{c^2} = k^2$ solution of wave equation

$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\frac{\omega}{c}\vec{A}_0 \sin(\vec{k}\cdot\vec{x} - \omega t)$

$\vec{B} = \vec{\nabla}\times\vec{A} \rightarrow -(\vec{k}\times\vec{A}_0)\sin(\vec{k}\cdot\vec{x} - \omega t)$

from the Coulomb Gauge Condition $\vec{\nabla}\cdot\vec{A} = 0$

$A_x = A_{0x}\cos(\vec{k}\cdot\vec{x} - \omega t)$

$\vec{\nabla}\cdot\vec{A} \rightarrow \frac{\partial A_x}{\partial x} = A_{0x} k_x \cos(\vec{k}\cdot\vec{x} - \omega t)$

$\vec{\nabla}\cdot\vec{A} = -\vec{k}\cdot\vec{A}_0 \sin(\vec{k}\cdot\vec{x} - \omega t) = 0$

because of Coulomb Gauge Condition $\vec{k}\cdot\vec{A}_0 = 0$ direction of the propagation of the wave

electric field $\propto \vec{A}_0$ magnetic field $\perp$ to both vectors $\vec{k}$ and $\vec{A}_0$

form the axis of a R.H.'d coordinate system

3 mutually perpendicular vectors.

What is the magnitude of the Poynting Vector $\vec{E}, \vec{B}$

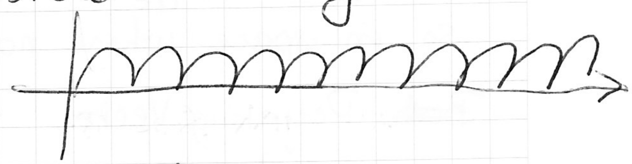
$|\vec{S}| = \frac{c}{4\pi}|\vec{E}||\vec{B}| = \frac{c}{4\pi}\left|\frac{\omega}{c}A_0\right|\sin(\vec{k}\cdot\vec{x} - \omega t)\,|kA_0|\sin(\vec{k}\cdot\vec{x} - \omega t)$

just writing the magnitudes $= \frac{c}{4\pi}\left(\frac{\omega}{c}\right)\left(\frac{\omega}{c}\right)A_0^2 \sin^2(\vec{k}\cdot\vec{x} - \omega t)$

Can't measure the instantaneous S value

visible $\omega \sim 10^{14}$ Hz

to many cycles per second



$\langle\vec{S}\rangle = I = \frac{c}{8\pi}\frac{\omega^2}{c^2}A_0^2 = \frac{\omega^2 A_0^2}{8\pi c}$ intensity of the wave

$$\boxed{\langle S\rangle = \frac{\omega^2 A_0^2}{8\pi c}}$$

$$H_0 = \frac{p^2}{2\mu} - \frac{Ze^2}{r}$$ hydrogen atom

$$\vec{p} = \vec{P} - \frac{q}{c}\vec{A} \qquad H_0$$

and add a term $q\phi$ - but don't need it (H eventually gives Lorenze Force.

$$\vec{H} = \frac{(\vec{p} + \frac{q}{c}\vec{A})^2}{2\mu} - \frac{Ze^2}{r} \qquad \vec{p}\cdot\vec{A} = \vec{A}\cdot\vec{p} \text{ in Coulomb Gauge.}$$

$$= \frac{p^2}{2\mu} + \underbrace{\frac{e}{\mu c}\vec{A}\cdot\vec{p}}_{} + \underbrace{\frac{e^2}{2\mu c^2}A^2}_{\vec{A}\text{ is small}} - \frac{Ze^2}{r}$$

So neglect in first approx.

$$H = H_0 + \frac{e}{\mu c}\vec{A}\cdot\vec{p} + \cancel{\frac{e^2}{2\mu c}A^2}^{\,0}$$

$$V = \frac{e}{\mu c}\vec{A}\cdot\vec{p} = \frac{e}{\mu c}\vec{A}_0\cos(\vec{k}\cdot\vec{x} - \omega t)\cdot\vec{p}$$

$$\vec{A} = \vec{A}_0\cos(\vec{k}\cdot\vec{x} - \omega t)$$ electromagnetic ~~wave~~ wave

$$V = \frac{e}{\mu c}\vec{A}_0\frac{1}{2}\underbrace{\left(e^{i(\vec{k}\cdot\vec{x} - \omega t)} + e^{-i(\vec{k}\cdot\vec{x} - \omega t)}\right)}_{\text{Can't have both at same time}}\cdot\vec{p}$$

$e^{i(\vec{k}\cdot\vec{x} - \omega t)}$ absorption

$e^{-i(\vec{k}\cdot\vec{x} - \omega t)}$ emission

Start with Absorption.

$$w_{fi}^{abs} = \frac{2\pi}{\hbar}\left|{}_0\langle f|V_0|i\rangle_0\right|^2 \delta(E_f^0 - E_i^0 - \hbar\omega)$$

$$V_0 = \frac{e}{2\mu c}e^{i\vec{k}\cdot\vec{x}}(\vec{A}_0\cdot\vec{p})$$ time independent part to perturbation

$${}_0\langle f|V_0|i\rangle_0$$

after = plane wave
Unbound state of Coulomb Hamiltonian

initial = ground state

$$\langle\vec{x}|i\rangle_0 = \sqrt{\frac{Z^3}{\pi a_0^3}}\, e^{-Zr/a_0}$$

$$\langle\vec{x}|f\rangle_0 = \sqrt{\frac{1}{2\pi\hbar}}\, e^{i\vec{p}\cdot\vec{x}/\hbar}$$

$$E_f^0 = \frac{p_f^2}{2\mu} \qquad \underbrace{E_i^0 = Z^2(Ry) = -Z^2 R_y = -Z^2(13.6\,eV)}_{\text{hydrogen-like ion}}$$

put $Z=1$ for hydrogen atom

$$ {}_0\langle f|V_0|i\rangle_0 = \frac{e}{2\mu c} \int \frac{1}{\sqrt{2\pi\hbar}} \sqrt{\frac{1}{\pi a_0^3}} \iiint_{r=0}^{\infty} e^{-(i/\hbar)\vec{p}_f\cdot\vec{x}}\, e^{i\vec{k}\cdot\vec{x}}\, \vec{A}_0 \frac{\hbar}{i}\vec{\nabla}\left(e^{-r/a_0}\right) \cancel{r^2 dr dz}\; r^2 dr \sin\theta d\theta d\phi $$

($Z=1$)

integrate by parts

$$ = \frac{-e}{2\mu c} \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\sqrt{\pi a_0^3}} \iiint \frac{\hbar}{i}\vec{\nabla}\left(e^{-i/\hbar(\vec{p}_f-\hbar\vec{k})\cdot\vec{x}}\right)\cdot\vec{A}e^{-r/a_0} r^2 dr\sin\theta d\theta d\phi $$

first term gives 0 because vanish at $r\to 0$ $r\to\infty$

$$ \langle f|\Omega|i\rangle = \langle i|\Omega^{+}|f\rangle^* = \langle i|\Omega|f\rangle^* = \langle \Omega f|i\rangle $$

$\hookrightarrow \langle i|\Omega^{+}|f\rangle$

Ω = hermitian

Ω^{+} is adjoint

$\Omega = \frac{\hbar}{i}\vec{\nabla}$

this is hermetian but $\vec{\nabla}$ alone is not

mathematically it is integration by parts

$$ \vec{\nabla}\left(e^{i\vec{p}'\cdot\vec{x}}\right) = i\vec{p}'e^{i\vec{p}'\cdot\vec{x}} $$

use this result

$$ {}_0\langle f|V_0|i\rangle_0 = \frac{-e}{2\mu c} \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\sqrt{\pi a_0^3}} \frac{\hbar}{i}\left(-\frac{i}{\hbar}\right) \iiint (\vec{p}_f-\hbar\vec{k})\cdot\vec{A}_0 e^{-i/\hbar(\vec{p}_f-\hbar\vec{k})\cdot\vec{x}}\, e^{-r/a_0} r^2 dr\sin\theta d\theta d\phi $$

$\vec{A}_0$ is a constant

$\vec{p}_f$ doesn't depend on the integration

final momentum of e^-

Coulomb Gauge

$$ = \frac{e}{2\mu c} \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\sqrt{\pi a_0^3}} \vec{p}_f\cdot\vec{A}_0 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{-i(\vec{p}_f-\hbar\vec{k})\cdot\vec{x}/\hbar}\, e^{-r/a_0} r^2 dr\sin\theta d\theta d\phi $$

take $\vec{p}_f-\hbar\vec{k}$ along $\hat{z}$ axis

$$ = \frac{e}{2\mu c} \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\sqrt{\pi a_0^3}} \vec{p}_f\cdot\vec{A}_0\; 2\pi \int_0^{\pi}\int_0^{\infty} e^{-i/\hbar|\vec{p}_f-\hbar\vec{k}|r\cos\theta}\, e^{-r/a_0} r^2 dr\sin\theta d\theta $$

$$ \int_0^{\pi} e^{-i|\vec{p}_f-\hbar\vec{k}|rx/\hbar}(-dx) \longrightarrow \int_{-1}^{+1} e^{-i/\hbar|\vec{p}_f-\hbar\vec{k}|xr} dx = $$

$$ \cancel{e^{-i/\hbar|\vec{p}_f-\hbar\vec{k}|xr}} \Longleftarrow \left.\frac{e^{-i/\hbar|\vec{p}_f-\hbar\vec{k}|xr}}{(-i/\hbar)|\vec{p}_f-\hbar\vec{k}|r}\right|_{-1}^{+1} $$

Substitute all that is left is the integration over r

fCl

$$ {}_0\langle f|V_0|i\rangle_0 = \frac{\pi e}{\mu c}\frac{1}{\cancel{\pi}}\frac{1}{\sqrt{2\hbar a_0^3}}\,\vec{p}_f\cdot\vec{A}_0\int_0^\infty\left[\frac{e^{-r\left(\frac{1}{a_0}+\frac{i}{\hbar}|\vec{p}_f-\hbar\vec{k}|\right)} - e^{-r\left(1/a_0-i|\vec{p}_f-\hbar\vec{k}|/\hbar\right)}}{-i/\hbar\,|\vec{p}_f-\hbar\vec{k}|\cancel{r}}\right]r^2dr $$

$P/\hbar$ is also 1/length

Gamma functions

$$ \int_0^\infty e^{-r(1/a_0+i/\hbar|\vec{p}_f-\hbar\vec{k}|)} $$

exponent can always be expanded in power series

$$ \int_0^\infty r e^{-\alpha r}dr = \int_0^\infty \frac{x e^{-x}dx}{\alpha^2} $$

$x=\alpha r$

$dx=\alpha dr$

$$ = \frac{e}{\mu c}\frac{1}{\sqrt{2\hbar a_0^3}}(\vec{p}_f\cdot\vec{A}_0)\frac{1}{-i/\hbar|\vec{p}_f-\hbar\vec{k}|}\left[\frac{1}{\left(\frac{1}{a_0}+\frac{i}{\hbar}|\vec{p}_f-\hbar\vec{k}|\right)^2}-\frac{1}{\left(1/a_0-i/\hbar|\vec{p}_f-\hbar\vec{k}|\right)^2}\right] $$

$$ \int_0 \frac{\Gamma(2)}{\alpha^2} = \frac{1}{\alpha^2} $$

Square this matrix element multiply by $\frac{2\pi}{\hbar}$

$$ \frac{d\sigma}{d\Omega} = \frac{1}{\text{area}} \quad \text{dimensions} $$

$H = H_0 + V$

prepare in H_0 ← no full hamiltonian won't remain in that state because of presense of V

transition probability

if V has time dependence, then can show prob transition rate is independent of time

Fermi's golden rule $V = V_0 e^{\pm i\omega t}$ ← em ← abs.

$$w_{\pm i} = \frac{2\pi}{\hbar} |{}_0\langle f|V|i\rangle_0|^2 \delta(E_f^{(0)} - E_i^{(0)} \pm \hbar\omega)$$

photoelectric
final state is plane wave

initial state is ground state of H

perturbation is interaction of incoming e/m wave with the electron $\vec{\nabla} \cdot \vec{A} = 0$

$\vec{p}_f' = \vec{p}_f - \hbar\vec{k}$ $\quad \vec{A}_0 \cdot \vec{k} = 0$ $\quad$ ~~$\vec{p}_f' \geq \vec{p}_f$~~

$${}_0\langle f|V_0|i\rangle_0 = \frac{8\pi N}{a_0} \frac{\vec{A}_0 \cdot \vec{p}_f}{\left[\frac{1}{a_0^2} + |p_f'|^2/\hbar^2\right]^2}$$

$$w_{fi} = \frac{2\pi}{\hbar}$$

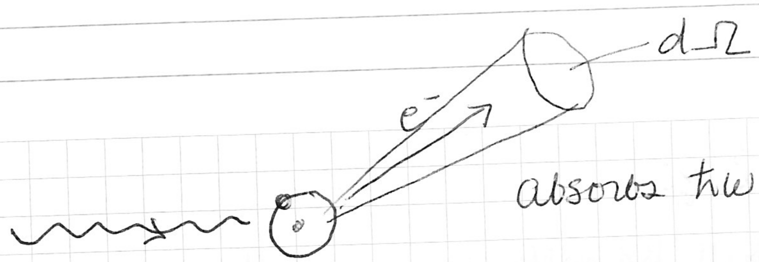
normalization constants in hydrogen

$$N = \frac{e}{2\mu c} \frac{1}{(2\pi\hbar)^{3/2}} \left(\frac{1}{\pi a_0^3}\right)^{1/2}$$

Calculate rate of transition

$$w_{fi} = \frac{2\pi}{\hbar} \frac{64\pi^2 N^2}{a_0^2} \frac{|\vec{A}_0 \cdot \vec{p}_f|^2}{\left[\frac{1}{a_0^2} + \frac{|\vec{p}_f'|^2}{\hbar^2}\right]^4} \delta\left(\frac{p_f^2}{2\mu} - E_i^{(0)} - \hbar\omega\right)$$

probability in some small angle
energy of e⁻



absorbs $\hbar\omega$

probability in $d\Omega$

how many final states e^- will have

$$d^3p_f = p_f^2\, dp_f\, d\Omega$$

$$\sum_{f,d\Omega} W_{fi} = \frac{128\pi^3}{\hbar a_0^2} d\Omega \int \frac{|\vec{A}_0\cdot\vec{p}_f|^2\, p_f^2\, dp_f\, \delta\left(\frac{p_f^2}{2\mu} - E_i^{(0)} - \hbar\omega\right) N^2}{\left[\frac{1}{a_0^2} + \frac{|\vec{p}_f'|^2}{\hbar^2}\right]^4}$$

can assume $\vec{A}_0\cdot\vec{p}_f$ is constant for a small interval of p_f

$$\frac{p_f^2}{2\mu} = E_i^{(0)} + \hbar\omega$$

$$p_f = \sqrt{2\mu(E_i^0 + \hbar\omega)}$$

$$\delta(f(x)) = \frac{1}{|f'(x)|_{x=x_0}}\,\delta(x-x_0)$$

$$f(x_0) = 0$$

$$f(p_f) = \frac{p_f^2}{2\mu} - E_i^{(0)} - \hbar\omega$$

$$f'(p_f) = \frac{p_f}{\mu}$$

$$\left[\frac{1}{(p_f/\mu)}\right]\left[\delta\left(p_f - \sqrt{2\mu(E_i^0 + \hbar\omega)}\right)\right]$$

$$\vec{p}_f' = \vec{p} - \hbar\vec{k}$$

$$\sum W_{fi} = \frac{128\pi^3\mu}{\hbar a_0^3} d\Omega \frac{|\vec{A}_0\cdot\vec{p}_f|^2\, p_f}{\left[\frac{1}{a_0^2} + \frac{|p_f'|^2}{\hbar^2}\right]^4}$$

probability in this $d\Omega$

energy absorbed by atom per unit time

$$\frac{dE_{abs}}{dt} = \int \frac{128\pi^3\mu\, 4|\vec{A}_0\cdot\vec{p}_f|^2\, p_f}{\hbar a_0^3 [\quad]} d\Omega\, \hbar\omega N^2$$

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

$$= \int \frac{128\pi^3\mu A_0^2\, p_f^3 \cos^2\theta}{\hbar a_0^3 \left[\frac{1}{a_0^2} + |p_f'|^2/\hbar^2\right]^4} d\Omega\, \hbar\omega N^2$$

$$= \left(\frac{4\pi}{3}\right) \frac{128\pi^3 \mu\omega A_0^2 \, p_f^3}{\hbar \, a_0^3 \left[1/a_0^2 + p_f'^2/\hbar^2\right]^4} \hbar\omega N^2$$

my head hurts

$$\cancel{\sigma} \cdot \frac{dE_{abs}}{dt} = \sigma I$$

previously derived

$$I = \frac{\omega^2}{8\pi c} A_0^2$$

$$\sigma = \frac{\frac{dE_{abs}}{dt}}{I}$$

energy / time

area

energy / time · area

$$\sigma = \frac{128\pi^3 \mu\omega \cancel{A_0^2} p_f^3 \frac{4\pi}{3} \hbar\omega \, 8\pi c}{\cancel{\hbar} a_0^3 \left[\frac{1}{a_0^2} + \frac{p_f'^2}{\hbar^2}\right]^4 \omega^2 \cancel{A_0^2}} \quad \frac{e^2}{4\mu^2 c^2} \frac{1}{(2\pi\hbar)^3} \frac{1}{\pi a_0^3}$$

$$\sigma = \frac{128\pi}{3} \frac{1}{\hbar^3\omega} \frac{p_f^3 a_0^3}{\left[1 + (p_f - \hbar k)^2 a_0^2/\hbar^2\right]^4} \frac{e^2}{\mu c}$$

dimensional analysis - area

$\hbar$ = action = momentum × length

$p_f a_0$ has diemention of $\hbar$

$$\frac{e^2}{\hbar c} = \frac{1}{137}$$

no dim

$$\Rightarrow \frac{e^2}{\mu c \omega} = \left(\frac{e^2}{\hbar c}\right) \frac{\hbar}{\mu\omega} = \frac{L(M L \cancel{T^{-1}})}{M \cancel{T^{-1}}} = L^2$$

Cross Section for effect.

either the atom can absorb ~~of~~ ~~e~~ ejected emit

if atom is excited $e^{+i\omega t}$ you get emission

transisio rate for stimillated emission + absoption are exactly the same for transition btwn to discrete states

---

Don't make approximations

excited states are not time independent

$e^{-\Gamma t/2\hbar}$ complicated theory not a delta function...

$$w_{fi} = \frac{2\pi}{\hbar} \left| {}_0\langle f | V_0 | i \rangle_0 \right|^2 \underbrace{\delta(E_f^0 - E_i^0 \pm \hbar\omega)}$$

this is replaced by

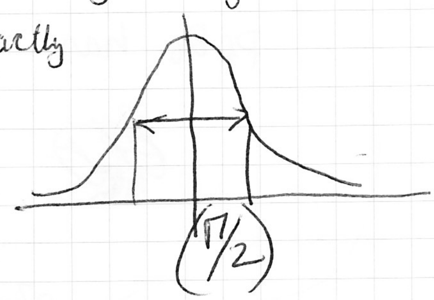
take into account finite lifetime of excited states

$$\frac{(\Gamma^2/2)}{[E_f^{(0)} - E_i^{(0)} - \hbar\omega]^2 + \Gamma^2/4}$$

Lorenzian function

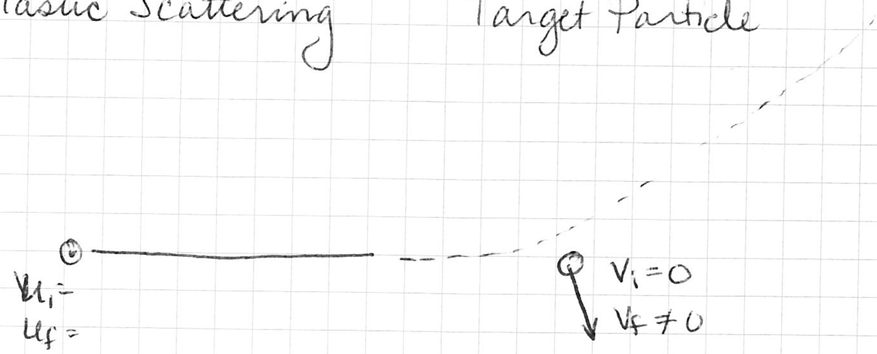
peak when they are exactly equal

emission



inelastic

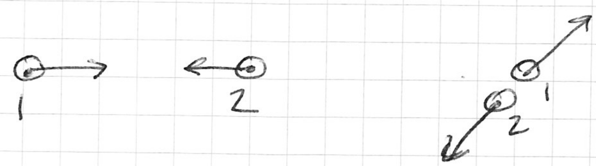
Elastic Scattering   Target Particle



lab frame

kinetic energy is conserved

simplify in COM frame $\Rightarrow$ the total momentum is zero



$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

time ind + time dep

non-deg
deg
pert i p

tdpt 2 problem   transition probabilities of some two states