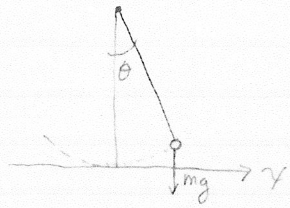
Ch. 5 take home test M. Webster

8.)



$\sin\theta = \frac{x}{l}$

the potential energy of the mass as a function of θ:

$U(\theta) = mgl(1-\cos\theta)$ potential on circular path with $U(0)=0$ by def

$U(\theta) = 2mgl(\sin^2 \theta/2)$ by trig ident.

taking the derivative with respect to θ:

$dU = \cancel{4}^{2} mgl\sin(\theta/2)\cos(\theta/2)\left(\frac{1}{\cancel{2}}\right)d\theta$

$dU = mgl\sin\theta\, d\theta$ by trig. ident.

dividing by dx

$$\frac{dU}{dx} = mgl\sin\theta\frac{d\theta}{dx}$$

substituting $\sin\theta = \frac{x}{l}$

$$\frac{dU}{dx} = mgl\left(\frac{x}{l}\right)\frac{d}{dx}\left(\sin^{-1}\frac{x}{l}\right)$$

$$\frac{dU}{dx} = mgl\left(\frac{x}{l}\right)\left(\frac{1}{l\sqrt{1-\left(\frac{x}{l}\right)^2}}\right)$$

$$\frac{dU}{dx} = mg\left(\frac{x}{l}\right)\left(1-\left(\frac{x}{l}\right)^2\right)^{-1/2}$$

$$\frac{dU}{dx} = mg\left(\frac{x}{l}\right)\left(1+\left(\frac{-1}{2}\right)\left(\frac{x}{l}\right)^2+\left(\frac{3}{4}\right)\left(\frac{1}{2!}\right)\left(\frac{x}{l}\right)^4 \pm \cdots\right)$$ by binomial expansion

$$F(x) = -\frac{dU}{dx} \sim -mg\left(\frac{x}{l}\right)\left(1-\frac{1}{2}\frac{x^2}{l^2}\right)$$

$$F(x) = -mg\left(\frac{x}{l} - \frac{1}{2}\frac{x^3}{l^3}\right)$$ ← Force due to the gravitational field! How about the centripetal force

set $F(x) = ma = m\ddot{x}$

$$m\ddot{x} = -mg\left(\frac{x}{l} - \frac{1}{2}\frac{x^3}{l^3}\right)$$

$$\ddot{x} + \left(\frac{g}{l}\right)x - \left(\frac{g}{2l^3}\right)x^3 = 0$$ ✓

let $w_0^2 = \frac{g}{l}$ and $\varepsilon = \frac{g}{2l^3}$

$$\boxed{\ddot{x} + w_0^2 x - \varepsilon x^3 = 0}$$

the following solution is taken from the book!

to solve this non-linear differential equation, $\ddot{x} + w_0^2 x = \varepsilon x^3$; assume the solution has the form:

$x = x_0 + \varepsilon x_1$

substitute into equation:

$$\ddot{x}_0 + w_0^2 x_0 - \varepsilon x_0^3 + \varepsilon(\ddot{x}_1 + w_0^2 x_1 - \varepsilon x_1^3) = 0$$

$$\ddot{x}_0 + w_0^2 x_0 + \varepsilon(\ddot{x}_1 + w_0^2 x_1 - x_0^3) + \varepsilon^2(-x_1^3) = 0$$

equating like powers of ε to zero:

$$\ddot{x}_0 + w_0^2 x_0 = 0$$

$$(D^2 + w_0^2)x_0 = 0 \Rightarrow \lambda = \pm i w_0$$

$$x_0 = A_0 e^{iw_0 t} + Be^{-w_0 t}$$ Keeping real-valued functions

$$x_0 = A\cos w_0 t + B\sin w_0 t$$

$A = x_0(0)$ and $B = \dot{x}_0(0) = 0$ ← chose this initial cond also have..

$$\varepsilon(\ddot{x}_1 + \omega_0^2 x_1 - x_0^3) = 0$$

$$\ddot{x}_1 + \omega_0^2 x_1 = x_0^3$$

$$\ddot{x}_1 + \omega_0^2 x_1 = A^3 \cos^3 \omega_0 t$$

$$\ddot{x}_1 + \omega_0^2 x_1 = \frac{1}{4} A^3 (3\cos\omega_0 t + \cos 3\omega_0 t)$$ by trig. ident.

the solution is:

$$x_1(t) = \frac{A^3}{32\omega_0^2}(12\omega_0 t \sin\omega_0 t - \cos 3\omega_0 t)$$

now to get rid of those terrible secular terms:

start with the equation of motion:

$$\ddot{x} + \omega_0^2 x - \varepsilon x^3 = 0$$

expanding the solution

$$x(t) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots = \sum_{n=0}^{\infty} (\varepsilon)^n x_n$$

expand also the square of the frequency:

$$\omega_0^2 = \omega^2 + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2 + \cdots = \omega^2 + \sum_{n=1}^{\infty} \alpha_n (\varepsilon)^n$$

now determine $\alpha_1, \alpha_2, \ldots \alpha_n$ so that the secular terms are eliminated

substitute into the differential equation of motion:

$$\sum_{n=0}^{\infty} (\varepsilon)^n \ddot{x}_n + \left(\omega^2 + \sum_{n=1}^{\infty} \alpha_n (\varepsilon^n)\right)\left(\sum_{n=0}^{\infty} (\varepsilon)^n x_n\right) - \left(\sum_{n=0}^{\infty} (\varepsilon)^{n+1} x_n\right) = 0$$

rearranging:

$$\sum_{n=0}^{\infty} (\varepsilon)^n \left[\ddot{x}_n + w^2 x_n \right] - \sum_{n=0}^{\infty} (\varepsilon)^{n+1} x_n + \sum_{n=1}^{\infty} \alpha_n (\varepsilon)^n \sum_{n=0}^{\infty} (\varepsilon)^n x_n = 0$$

equate like powers of ε to zero:

$$\ddot{x}_0 + w^2 x = 0$$

$$\ddot{x}_1 + w^2 x_1 = x_0^3 - \alpha_1 x_0$$

$$\ddot{x}_2 + w^2 x_2 = 3x_0^2 x_1 - \alpha_1 x_1 - \alpha_2 x_0$$

chose initial conditions:

$x(0) = A$ $\qquad \dot{x}(0) = 0$

$x_0(0) = A$ $\qquad \dot{x}_0(0) = 0$

$x_i(0) = 0$ $\qquad \dot{x}_i(0) = 0$ $\qquad$ for $i = 1, 2, 3, \ldots n$

since $x_0 = A\cos wt$

substitute into $\ddot{x}_1 + w^2 x_1 = x_0^3 - \alpha_1 x_0$

$$\ddot{x}_1 + w^2 x_1 = A^3 \cos^3 wt - \alpha_1 A \cos wt$$

$$\ddot{x}_1 + w^2 x_1 = \left(\tfrac{3}{4} A^3 - \alpha_1 A\right) \cos wt + \tfrac{1}{4} A^3 \cos 3wt$$

since the coefficient of $\cos wt$ must be zero since it is a soln. of the homogeneos eq:

$$\alpha_1 = \tfrac{3}{4} A^2$$

$$\ddot{x}_1 + w^2 x_1 = \tfrac{1}{4} A^3 \cos 3wt$$

$$\ddot{x}_1 + \omega^2 x_1 = \tfrac{1}{4} A^3 \cos 3\omega t$$

$$x_1(t) = \frac{A^3}{32\omega^2}(\cos\omega t - \cos 3\omega t)$$ satisfies initial conditions

and

$$\omega_0 = (\omega^2 + \alpha\varepsilon)^{1/2} = \left(\omega^2 + \tfrac{3}{4}A^2\varepsilon\right)^{1/2}$$

thus correct to first order:

$$\boxed{\omega \sim \omega_0\left(1 - \frac{3A^2}{8\omega_0^2}\varepsilon\right)}$$ relates frequency and amplitude

$$\boxed{x(t) = \left(1 + \frac{\varepsilon A^2}{32\omega^2}\right)A\cos\omega t + \left(\frac{-A^3}{32\omega^2}\right)A\cos 3\omega t}$$

first-order solution to Duffing equation

12.) particle of mass m

$$F(x) = m\ddot{x} \begin{cases} +F_0 & x<0 \\ -F_0 & x>0 \end{cases} \qquad \text{thus} \qquad \ddot{x} = \begin{cases} +F_0/m & x<0 \\ -F_0/m & x>0 \end{cases}$$

first consider $x<0$

$$\ddot{x} = (F_0/m)$$

$$\ddot{x} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

$$v\,dv = \left(\frac{F_0}{m}\right)dx$$

$$\frac{v^2}{2} = \frac{F_0}{m}x + C \qquad \dot{x}(x=-A) = 0$$

$$0 = \frac{v^2}{2} = \frac{F_0}{m}(-A) = C \qquad \therefore \quad C = \frac{F_0 A}{m}$$

$$\frac{v^2}{2} = \frac{F_0}{m}(x) + \frac{F_0}{m}A$$

$$v^2 = \frac{2F_0}{m}(x+A) \qquad \text{for } x<0$$

now consider $x>0$

$$\ddot{x} = (-F_0/m)$$

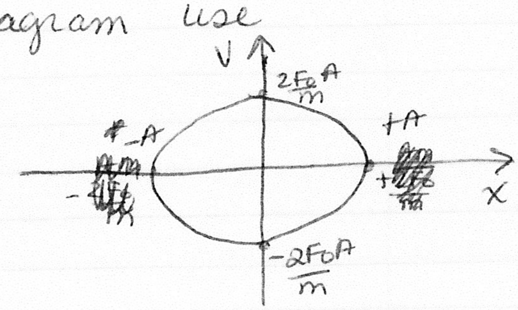
$$\frac{v^2}{2} = -\frac{F_0}{m}x + C' \qquad \dot{x}(x=+A) = 0$$

$$0 = -\frac{F_0}{m}A + C' \qquad \therefore \quad C' = \frac{F_0}{m}A$$

$$v^2 = \frac{2F_0}{m}(-x+A)$$

thus to graph the phase diagram use

$$\begin{cases} v^2 = \frac{2F_0}{m}(x+A) & \text{for } x<0 \\ v^2 = \frac{2F_0}{m}(-x+A) & \text{for } x>0 \end{cases}$$



assumes that the energy of the particle $E > U(x) \Rightarrow E > \pm F_0 x$

to calculate the period of motion in terms of m, F_0, and the amplitude A (disregard damping)

calculate time for particle to go from 0 to $+A$ and multiply by 4:

$$\ddot{x} = -F_0/m \qquad x > 0$$

$$\dot{x} = -\frac{F_0}{m}t + C_1 \qquad \dot{x}(0) = 0 \quad \therefore C_1 = 0$$

$$x = -\left(\frac{F_0}{2m}\right)t^2 + C_2 \qquad x(0) = A \quad \therefore C_2 = A$$

equation of motion in time is.

$$x = A - \frac{F_0}{2m}t^2$$

solving for t

$$t = \left[\frac{2m}{F_0}(A - x)\right]^{1/2}$$

time to go from 0 to A:

$$\Delta t = \left|\left[\frac{2m}{F_0}(A - 0)\right]^{1/2} - \left[\frac{2m}{F_0}(A - A)\right]^{1/2}\right|$$

$$\Delta t = \left(\frac{2mA}{F_0}\right)^{1/2}$$

thus the total period to go all around is $\Delta t(0 \to A) + \Delta t(A \to 0) + \Delta t(0 \to -A) + \Delta t(-A \to 0)$

$$T = 4\Delta T = 4\left(\frac{2mA}{F_0}\right)^{1/2} = \left(\frac{32mA}{F_0}\right)^{1/2}$$