Ch. 4 homework. Meg Webster

good! $\frac{57}{60}$

2.) lightly damped

$E = \frac{1}{2} k A_0^2$ total energy

$E = \frac{1}{2} k (A_0^2 - A_1^2)$ energy loss during one period

$A_1 = A_0 e^{-\beta\tau}$ τ = one period = $\frac{2\pi}{\omega_R}$

$$2\pi \left(\frac{\text{total energy}}{\text{energy loss during one period}} \right) = \frac{2\pi \left(\frac{1}{2} k A_0^2 \right)}{\frac{1}{2} k \left(A_0^2 - A_0^2 e^{-2\beta\tau} \right)}$$

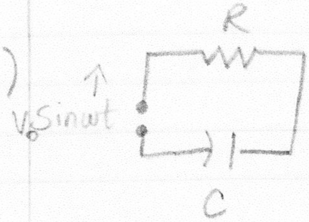
$$= 2\pi \left(\frac{1}{1 - e^{-4\pi\beta/\omega_R}} \right) = 2\pi \left(1 - e^{-4\pi\beta/\omega_R} \right)^{-1}$$

$$= 2\pi \left(1 - \left(1 - \frac{4\pi\beta}{\omega_R} + \frac{16\beta^2\pi^2}{\omega_R^2} + \dots \right) \right)^{-1}$$

$$\sim 2\pi \left(\frac{4\pi\beta}{\omega_R} \right)^{-1}$$

$$\sim \frac{\omega_R}{2\beta} \quad ✓$$

$$\sim Q$$

5.) 

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

$$\cancel{L\ddot{q}}^{\,0} + R\dot{q} + \frac{q}{C} = \mathcal{E}_0 \sin\omega t$$

$$R\dot{I} + C^{-1}I = \omega_0 \mathcal{E}_0 \cos\omega_0 t$$

homog. soln.:

$$\dot{I} + \frac{I}{RC} = 0$$

$$\left(D + \frac{1}{RC}\right)I = 0 \qquad D = -\frac{1}{RC} = \tau$$

$$I_H = c_0 e^{-\tau t}$$

assume as a particular soln:

$$I_p = C_1 \sin\omega_0 t + C_2 \cos\omega_0 t$$

thus substituting

$$R(\omega_0 C_1 \cos\omega_0 t - C_2 \omega_0 \sin\omega_0 t) + C^{-1}(C_1 \sin\omega_0 t + C_2 \cos\omega_0 t) = \omega_0 \mathcal{E}_0 \cos\omega_0 t$$

equating sines and cosines:

$$R\omega_0 C_1 + C_2 C^{-1} = \omega_0 \mathcal{E}_0 \qquad -R\omega_0 C_2 + C^{-1}C_1 = 0$$

$$C_1 = (CR\omega_0)C_2$$

$$R\omega_0(CR\omega_0)C_2 + C^{-1}C_2 = \omega_0 \mathcal{E}_0$$

$$C_2 = \frac{\omega_0 \mathcal{E}_0 (C^{-1})}{(R\omega_0)^2 + 1} \qquad C_1 = \frac{\mathcal{E}_0 R \omega_0^2}{(R\omega_0)^2 + 1}$$

thus $I_{gen} = I_H + I_p$

$$I = C_0 e^{-\tau t} + \left(\frac{\mathcal{E}_0 R \omega_0^2}{(R\omega_0)^2 + 1}\right) \sin \omega_0 t + \left(\frac{\omega_0 \mathcal{E}_0 C^{-1}}{(R\omega_0)^2 + 1}\right) \cos \omega_0 t$$

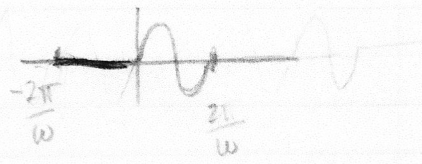
lim $\omega_0 \to 0$

$$I = C_0 e^{-\tau t} + 0 + 0$$

thus the particular or steady state solution $I_p \to 0$ as $\omega_0 \to 0$

the frequency of the source of alternating emf. ✓

13.) $F(t) = \begin{cases} 0 & -2\pi/\omega < t < 0 \\ \sin\omega t & 0 < t < 2\pi/\omega \end{cases}$



$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt = 0$$

$$a_n = \frac{2}{T}\int_0^T F(t')\cos n\omega t'\,dt' = 0 \quad \text{for} \quad F(t') = 0$$

$$a_n = \frac{2}{T}\int_0^T \sin\omega t' \cos n\omega t'\,dt' = \frac{2\omega_F}{2\pi}\int_0^{2\pi/\omega} \sin\omega_0 t \cos n\omega_F t\,dt \qquad \omega_0 = \frac{\omega_F}{2}$$

$$= \frac{\omega_n}{\pi}\int_0^{2\pi/\omega_n} \sin \omega_n t' \cos\left(\frac{n}{2}\right)\omega_n t'\,dt'$$

$$a_n = \frac{\omega_n}{\pi}\int_0^{2\pi/\omega_n} \sin \omega_n t' \cos\left(\frac{n}{2}\omega_n t'\right)dt'$$

$$a_n = -\frac{\omega_n}{2\pi}\left[\frac{\cos\left(\omega_n - \frac{n}{2}\omega_n\right)t'}{2\omega_n - n\omega_n} + \frac{\cos\left(\omega_n + \frac{n}{2}\omega_n\right)t'}{2\omega_n + n\omega_n}\right]\Bigg|_0^{2\pi/\omega_n}$$

$$a_n = -\frac{\omega_n}{2\pi}\left[\frac{\cos\left(\omega_n\left(1-\frac{n}{2}\right)2\pi/\omega_n\right)}{\omega_n\left(1-\frac{n}{2}\right)} + \frac{\cos\left(\omega_n\left(1+\frac{n}{2}\right)2\pi/\omega_n\right.}{\omega_n\left(1+\frac{n}{2}\right)} - \frac{1}{\omega_n\left(1-\frac{n}{2}\right)} - \frac{1}{\omega_n(1+\right.$$

$$a_n = -\frac{1}{2\pi}\left[\frac{\cos\left(2\pi\left(1-\frac{n}{2}\right)\right)}{\left(1-\frac{n}{2}\right)} + \frac{\cos\left(2\pi\left(1+\frac{n}{2}\right)\right)}{\left(1+\frac{n}{2}\right)} - \frac{1}{\left(1-\frac{n}{2}\right)} - \frac{1}{\left(1+\frac{n}{2}\right)}\right.$$

for $n = 2, 4, \ldots\ 2m \qquad a_n = 0$
for $n = 1$

$$a_1 = -\frac{1}{\pi}\left[\frac{\cos\pi}{1} + \frac{\cos 3\pi}{3} - \frac{1}{1} - \frac{1}{3}\right] = \frac{8}{3\pi}$$

$$a_{2m+1} = -\frac{1}{2\pi}\left[\frac{-2}{\left(1-\left(\frac{2m+1}{2}\right)\right)} - \frac{2}{\left(1+\left(\frac{2m+1}{2}\right)\right)}\right] = \frac{1}{\pi}\left[\frac{1}{1-\frac{n}{2}} + \frac{1}{1+\frac{n}{2}}\right] = \frac{1+\frac{n}{2}+1-\frac{n}{2}}{1-\left(\frac{n}{2}\right)^2}\left(\frac{1}{\pi}\right) = \frac{2}{\pi\left(1-\left(\frac{n}{2}\right)^2\right)}$$

$$b_n = \frac{\omega_n}{\pi}\int_0^{2\pi/\omega_n} \sin\omega_n t' \sin\frac{n}{2}\omega_n t' \, dt'$$

$$b_n = \frac{\omega_n}{\pi}\left[\frac{\sin(\omega_n - \frac{n}{2}\omega_n)t}{2(\omega_n - \frac{n}{2}\omega_n)} - \frac{\sin(\omega_n + \frac{n}{2}\omega_n)t}{2(\omega_n - \frac{n}{2}\omega_n)}\right]\Bigg|_0^{2\pi/\omega_n}$$

$$b_n = \frac{1}{2\pi}\left[\frac{\sin(1-\frac{n}{2})2\pi}{(1-\frac{n}{2})} - \frac{\sin 2\pi(1+\frac{n}{2})}{(1+\frac{n}{2})}\right]$$

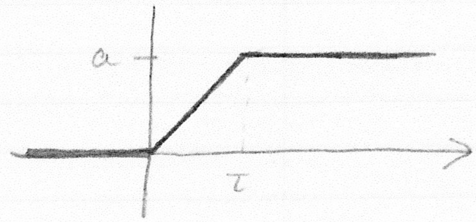
for all n $\quad b_n = 0$ ✓

substituting into Fourier equation:
odd valued $n = 2m+1$

$$F(t) = \sum_{m=1}^{\infty}\left(\frac{2}{\pi\left(1-\left(\frac{2m+1}{2}\right)^2\right)}\right)\cos(2m+1)\omega t$$

15.) damped linear oscillator, originally in its equilibrium position, is subjected to a forcing function

$$\frac{F(t)}{m} = \begin{cases} 0 & t>0 \\ \left(\frac{a}{\tau}\right)t & 0<t<\tau \\ a & t<\tau \end{cases}$$



$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F(t)$$

Using Laplace transformations

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = a\left(\frac{t}{\tau}\right)$$

$$\int_0^\infty e^{-pt}\ddot{x}\,dt + 2\beta\int_0^\infty e^{-pt}\dot{x}\,dt + \omega_0^2\int_0^\infty e^{-pt}x\,dt = \frac{a}{\tau}\int_0^\infty te^{-pt}dt$$

$$p^2 f(p) + 2\beta p f(p) + \omega_0^2 f(p) = \frac{a}{\tau}\left(\frac{1}{p^2}\right)$$

$$f(p) = \frac{a}{\tau p^2}\frac{1}{(p^2+2\beta p+\omega_0^2)}$$

$$f(p) = \frac{a}{\tau}\left[\frac{1}{p^2(p^2+2\beta p+\omega_0^2)}\right]$$

$$f(p) = \frac{a}{\tau\omega_0^2}\left(\frac{1}{p}\right)\left[\frac{1}{p} - \frac{p+\beta}{(p+\beta)^2+\omega_1^2} - \frac{\beta}{\omega_1}\frac{\omega_1}{(p+\beta)^2+\omega_1^2}\right]$$

$$f(p) = \frac{a}{\tau\omega_0^2}\left[\frac{1}{p^2} - \frac{1}{(p+\beta)^2+\omega_1^2} - \frac{2\beta}{p[(p+\beta)^2+\omega_1^2]}\right]$$

$$f(p) = \frac{a}{\tau\omega_0^2}\left[\frac{1}{p^2} - \frac{1}{\omega_1}\left(\frac{\omega_1}{(p+\beta)^2+\omega_1^2}\right) - \left(\frac{2\beta}{\beta^2+\omega_1^2}\right)\left(\frac{1}{p} - \frac{p+2\beta}{(p^2+2\beta p+\beta^2+\omega_1^2)}\right)\right.$$

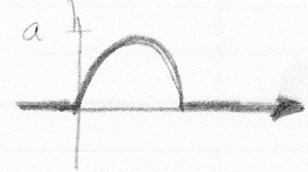
$$f(p) = \frac{a}{\tau \omega_0^2}\Big[\left(\frac{-2\beta}{\beta^2+\omega_1^2}\right)\frac{1}{p} + \frac{1}{p^2} - \left(\frac{1}{\omega_1}\right)\left(\frac{\omega_1}{(p+\beta)^2+\omega_1^2}\right) + \frac{2\beta}{(\beta^2+\omega_1^2)}\left(\frac{p+2\beta}{p^2+2\beta p+\beta^2+\omega_1^2}\right.$$

taking the inverse Laplace transform:

$$F(t) = x(t) = \frac{a}{\tau \omega_0^2}\Big[\frac{-2\beta}{(\beta^2+\omega_1^2)} + \left(t\right) - \left(\frac{1}{\omega_1}\right)e^{-\beta t}\sin\omega_1 t + \frac{2\beta}{(\beta^2+\omega_1^2)}e^{-\beta t}\cos(\beta^2+\omega_1^2$$

for $0 < t < \tau$

19.) $\frac{F(t)}{m} = \begin{cases} 0 & t<0 \\ a\sin\omega t & 0 < t < \pi/\omega \\ 0 & t > \pi/\omega \end{cases}$



$$x(t) = \int_0^{\pi/\omega} \frac{m}{m\omega_1} a \sin\omega t' e^{-\beta(t-t')} \sin\omega_1 (t-t') dt'$$

$$x(t) = \int_0^{\pi/\omega} \frac{m}{m\omega_1} a\, e^{-i\omega t'} e^{-\beta(t-t')} e^{i\omega_1(t-t')} dt' \qquad z = \omega_1(t-t')$$

$$x(t) = \int_0^{\omega_1 t} \frac{ma}{m\omega_1^2} e^{-i\omega t} e^{[(r-\beta)/\omega_1]z} \sin z\, dz$$

$$x(t) = \frac{a}{(\omega^2-\beta^2)+\omega_1^2}\left[e^{-i\omega t} - e^{-\beta t}\left(\cos\omega_1 t - \frac{\omega-\beta}{\omega_1}\sin\omega_1 t\right)\right]$$

$$x(t) = \frac{a}{(\omega^2-\beta^2)+\omega_1^2}\left[\sin\omega t - e^{-\beta t}\left(\cos\omega_1 t - \frac{\omega-\beta}{\omega_1}\sin\omega_1 t\right)\right]$$