1. Consider the motion in one dimension of a particle subjected to potential V=Γ|x| (where Γ=constant). Use action-angle variables to find the period of the motion as a function of energy.

Solution:

The kinetic energy of the particle is given by:

$$T = \frac{1}{2}m\dot{x}^2$$

The potential energy of the particle is given by:

$$V = \Gamma\left|x\right|$$

The Energy of the particle is given by:

$$E = T + V = \frac{1}{2}m\dot{x}^2 + \Gamma\left|x\right|$$

The Lagrangian of the system is given by:

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \Gamma\left|x\right|$$

The conjugate momenta is given by:

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

The Hamiltonian is given by:

$$H = p\dot{x} - L = m\dot{x} - T + V = m\dot{x}\dot{x} - \frac{1}{2}m\dot{x}^2 + \Gamma\left|x\right| = \frac{p^2}{2m} + \Gamma\left|x\right| = E$$

The conjugate time derivative of momenta and spatial coordinates are:

$$\dot{x} = -\frac{\partial H}{\partial p} = -\frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial \Gamma\left|x\right|}{\partial x}$$

Algebraically, solve for momentum in terms of E and x.

$$p = \pm\sqrt{2m\left[E - \Gamma\left|x\right|\right]}$$

The time independent Hamilton-Jacobi generating function as a function of x is given by (eq. 10.17):

$$W = \int_0^x p dx = \pm \int_0^x \sqrt{2m\left[E - \Gamma |x|\right]} dx$$

The action is given by (the factor of 4 multiplied by the integral of the pos-x,pos-p quadrant because of symmetry) (eq. 10.82):

$$J = \frac{1}{2\pi} \oint p dx = \frac{2}{\pi} \int_0^{x\max} \sqrt{2m\left[E - \Gamma |x|\right]} dx = \frac{2}{\pi} \sqrt{2mE} \int_0^{x\max} \sqrt{1 - \Gamma |x| / E} dx$$

Since the x is only in the positive x region:

$$J = \frac{2}{\pi} \sqrt{2mE} \int_0^{x\max} \sqrt{1 - \Gamma |x| / E} dx$$

$$J = \frac{2}{\pi} \sqrt{2mE} \left(\frac{2E}{3\Gamma} \right) \left(1 - \Gamma x / E\right)^{3/2} \Big|_0^{x\max} = \frac{2}{\pi} \sqrt{2mE} \left(\frac{2E}{3\Gamma} \right) \left(1 - \Gamma x_{\max} / E\right)^{3/2} - \frac{2}{\pi} \sqrt{2mE} \left(\frac{2E}{3\Gamma} \right)$$

Since the particle is changing direction at xmax then it's kinetic energy becomes zero instantaneously. Therefore $\Gamma x_{\max} = E$

$$J = \frac{4}{3\pi\Gamma} (2m)^{1/2} E^{3/2}$$

Algebraically express E in terms of J:

$$J = \frac{4\sqrt{2m}}{3\pi\Gamma} E^{3/2}$$

$$E^{3/2} = J \frac{3\pi\Gamma}{4\sqrt{2m}}$$

$$E = J^{2/3} \left(\frac{3\pi\Gamma}{4\sqrt{2m}} \right)^{2/3}$$

The prescribed solution calls for finding the period from the action angles (eqs 10.86, 10.87, 10.91).

$$\dot{w} = \frac{\partial H}{\partial J}, \quad w = vt + \beta,$$

$$E = J^{2/3}\left(\frac{3\pi\Gamma}{4\sqrt{2m}}\right)^{2/3} \Rightarrow \frac{1}{J^{1/3}} = \frac{1}{\sqrt{E}}\left(\frac{3\pi\Gamma}{4\sqrt{2m}}\right)^{1/3}$$

$$\dot{w} = \frac{\partial H}{\partial J} = \frac{2}{3}\left(\frac{3\pi\Gamma}{4\sqrt{2m}}\right)^{2/3} J^{-1/3} = \frac{2}{3}\left(\frac{3\pi\Gamma}{4\sqrt{2m}}\right)^{2/3} \frac{1}{\sqrt{E}}\left(\frac{3\pi\Gamma}{4\sqrt{2m}}\right)^{1/3} = \left(\frac{\pi\Gamma}{2\sqrt{2mE}}\right)$$

$$\dot{w} = \frac{dw}{dt} = \left(\frac{\pi\Gamma}{2\sqrt{2mE}}\right)$$

$$w = vt + \beta = \left(\frac{\pi\Gamma}{2\sqrt{2mE}}\right)t + \beta$$

$$v = \left(\frac{\pi\Gamma}{2\sqrt{2mE}}\right)$$

$$T = \frac{2\pi}{v} = \frac{2\pi}{\left(\frac{\pi\Gamma}{2\sqrt{2mE}}\right)} = \frac{4}{\Gamma}\left(2mE\right)^{1/2}$$

$$T = \frac{4\sqrt{2mE}}{\Gamma}$$

The frequency can also be found from:

$$\frac{1}{\omega} = \frac{\partial J}{\partial E} = \frac{d}{dE}\frac{4\sqrt{2m}}{3\pi\Gamma}E^{3/2} = \frac{3}{2}\frac{4\sqrt{2m}}{3\pi\Gamma}E^{1/2} = \frac{2\sqrt{2mE}}{\pi\Gamma}$$

The period is given by:

$$T = \frac{2\pi}{\omega} = 2\pi\left(\frac{2\sqrt{2mE}}{\pi\Gamma}\right) = \frac{4\sqrt{2mE}}{\Gamma}$$

2.0 Determine the Lagrangian density for a three-dimensional sound wave in air. Determine the pressure, density, velocity, and thermal waves for a plane wave and a radiating point source.

Reference: Elmore & Heald "Physics of Waves" and
http://en.wikiversity.org/wiki/Advanced_Classical_Mechanics/Continuum_Mechanics

Density of air: $\rho_0 = 1.29 \text{ kg/m}^3$

In considering volume changes in an elastic medium, we can write Hooke's Law:

$$
\begin{aligned}
&P = -B\sigma \\
&P = \text{hydrostatic pressure} \\
&-\sigma = \text{strain} \\
&B = \text{bulk modulus [force/area]}
\end{aligned}
$$

The volume strain associated with incremental pressure is related to the displacement (vector function of position and time):

$$
\begin{aligned}
&\sigma = \frac{\Delta V}{V} = \nabla \cdot \vec{\wp} \\
&P = -B\sigma = -B\nabla \cdot \vec{\wp} \\
&\vec{\wp} = \text{displacement vector function} \\
&P = \text{Pressure}
\end{aligned}
$$

The incremental changes in pressure cause small changes ΔV of volume V (equilibrium volume V) the work done is:

$$W = -\tfrac{1}{2} P\Delta V$$

The potential energy density is:

$$U = \frac{W}{V} = -\tfrac{1}{2} P \frac{\Delta V}{V} = -\tfrac{1}{2} P\theta = \tfrac{1}{2} B\theta^2 = \frac{P^2}{2B} = \frac{P^2}{2\rho_0 v^2}$$

From the net vector force on a cubical volume element a wave equation for pressure is found:

$$d\vec{F} = -\nabla P dxdydz = \underbrace{\rho_0 dxdydz}_{\substack{\text{mass of volume}\\ \text{element}}} \underbrace{\frac{\partial^2 \vec{\wp}}{\partial t^2}}_{\substack{\text{acceleration}\\ \text{of volume}\\ \text{element}}}$$

$$
\begin{aligned}
&-\nabla P = \rho_0 \frac{\partial^2 \vec{\wp}}{\partial t^2} \Rightarrow -\nabla \cdot \nabla P = \rho_0 \frac{\partial^2 \nabla \vec{\wp}}{\partial t^2} \Rightarrow \nabla^2 P = \frac{\rho_0}{B} \frac{\partial^2 P}{\partial t^2} = \frac{1}{v^2} \frac{\partial^2 P}{\partial t^2} \\
&v = \sqrt{B/\rho_0} = \text{the velocity of the wave}
\end{aligned}
$$

This is shown to result in a sinusoidal wave of the displacement vector:

$$\vec{\wp}(\vec{r},t) = \frac{1}{\rho_0 \omega^2} \nabla P_\omega(\vec{r}) e^{-i\omega t}$$
$$\nabla^2 P_\omega + \frac{\omega^2}{v^2} P_\omega = 0 \Rightarrow P(\vec{r},t) = P_\omega(\vec{r}) e^{-i\omega t}$$
$$P(\vec{r},t) = P_{\max} e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \text{ just set the direction to the x axis}$$
$$P(x,t) = P_{\max} e^{-i(kx-\omega t)}$$

The kinetic energy density of the wave based on the above is:

$$T = \tfrac{1}{2}\rho_0 \left[\mathrm{Re}\left(\frac{\partial \vec{\wp}}{\partial t} \right) \right]^2$$

From the kinetic energy density and the potential energy density the Lagrangian density is found as:

$$L = \tfrac{1}{2}\rho_0 \left[\mathrm{Re}\left(\frac{\partial \vec{\wp}}{\partial t} \right) \right]^2 + \frac{(\nabla \cdot \vec{\wp})^2}{2\rho_0 v^2}$$

If the gas is expanded adiabatically then:

$$B = \gamma P$$
$$B = \text{adiabatic bulk modulus [force/area]}$$
$$\gamma = C_P / C_V$$
$$C_P = \text{heat capacity at constant pressure}$$
$$C_V = \text{heat capacity at constant volume}$$

So the state variables (pressure and thermal properties) of the air can be related to the velocity of sound in the air:

$$P = \frac{v^2 \rho_0}{\gamma} = \text{Pressure}$$
$$T = \frac{v^2 m_0}{\gamma k_B} = \text{Temperature}$$
$$m_0 = \text{mass of individual molecule}$$
$$k_B = \text{Boltzmann constant}$$

For a radiating point source, the wave equation becomes (and solution for the pressure wave):

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\left(\frac{\partial P}{\partial r}\right)-\frac{1}{v^2}\frac{\partial^2 P}{\partial t^2}=0$$

$$P=\frac{A}{r}e^{i(kr-\omega t)}$$

3. Take ψ and $\psi*$ as independent field variables in the following Lagrangian density and determine the wave equation. What is it? What are the canonical momenta? (I hope that's the same thing as conjugate momenta!)

$$L_{Density} = \frac{\hbar^2}{2m}\nabla\psi\cdot\nabla\psi* + V\psi*\psi + \frac{\hbar}{2i}\left(\psi*\dot{\psi} - \psi\dot{\psi}*\right)$$

Solution:

Substitute into the Euler-Lagrange (for Lagrange Density) equations:

$$\frac{\partial L_{Density}}{\partial\psi} - \partial_x\frac{\partial L_{Density}}{\partial(\partial_x\psi)} - \partial_y\frac{\partial L_{Density}}{\partial(\partial_y\psi)} - \partial_z\frac{\partial L_{Density}}{\partial(\partial_z\psi)} - \partial_t\frac{\partial L_{Density}}{\partial(\partial_t\psi)} = 0$$

$$\frac{\partial L_{Density}}{\partial\psi*} - \partial_x\frac{\partial L_{Density}}{\partial(\partial_x\psi*)} - \partial_y\frac{\partial L_{Density}}{\partial(\partial_y\psi*)} - \partial_z\frac{\partial L_{Density}}{\partial(\partial_z\psi*)} - \partial_t\frac{\partial L_{Density}}{\partial(\partial_t\psi*)} = 0$$

First do $\psi*$

$$\frac{\partial L_{Density}}{\partial\psi*} = \frac{\partial}{\partial\psi*}\left[\frac{\hbar^2}{2m}\nabla\psi\cdot\nabla\psi* + V\psi*\psi + \frac{\hbar}{2i}\left(\psi*\dot{\psi} - \psi*\dot{\psi}\right)\right] = \left[V\psi + \frac{\hbar}{2i}\dot{\psi}\right]$$

$$\partial_x\frac{\partial L_{Density}}{\partial(\partial_x\psi*)} = \partial_x\frac{\partial L_{Density}}{\partial(\partial_x\psi*)}\left[\frac{\hbar^2}{2m}\left(\partial_x\psi\partial_x\psi* + \partial_y\psi\partial_y\psi* + \partial_z\psi\partial_z\psi*\right) + V\psi*\psi + \frac{\hbar}{2i}\left(\psi*\dot{\psi} - \psi\dot{\psi}*\right)\right]$$

$$\partial_x\frac{\partial L_{Density}}{\partial(\partial_x\psi*)} = \partial_x\frac{\partial_x\psi\hbar^2}{2m} = \frac{\hbar^2}{2m}\partial_x^2\psi$$

$$\partial_y\frac{\partial L_{Density}}{\partial(\partial_y\psi*)} = \partial_y\frac{\partial L_{Density}}{\partial(\partial_y\psi*)}\left[\frac{\hbar^2}{2m}\left(\partial_x\psi\partial_x\psi* + \partial_y\psi\partial_y\psi* + \partial_z\psi\partial_z\psi*\right) + V\psi*\psi + \frac{\hbar}{2i}\left(\psi*\dot{\psi} - \psi\dot{\psi}*\right)\right]$$

$$\partial_y\frac{\partial L_{Density}}{\partial(\partial_y\psi*)} = \partial_y\frac{\partial_y\psi\hbar^2}{2m} = \frac{\hbar^2}{2m}\partial_y^2\psi$$

$$\partial_z\frac{\partial L_{Density}}{\partial(\partial_z\psi*)} = \partial_z\frac{\partial L_{Density}}{\partial(\partial_z\psi*)}\left[\frac{\hbar^2}{2m}\left(\partial_x\psi\partial_x\psi* + \partial_y\psi\partial_y\psi* + \partial_z\psi\partial_z\psi*\right) + V\psi*\psi + \frac{\hbar}{2i}\left(\psi*\dot{\psi} - \psi\dot{\psi}*\right)\right]$$

$$\partial_z\frac{\partial L_{Density}}{\partial(\partial_z\psi*)} = \partial_z\frac{\partial_z\psi\hbar^2}{2m} = \frac{\hbar^2}{2m}\partial_z^2\psi$$

$$\partial_t \frac{\partial L_{Density}}{\partial(\partial_t \psi^*)} = \partial_t \frac{\partial L_{Density}}{\partial(\partial_t \psi^*)}\left[\frac{\hbar^2}{2m}\left(\partial_x\psi\partial_x\psi^* + \partial_y\psi\partial_y\psi^* + \partial_z\psi\partial_z\psi^*\right) + V\psi^*\psi + \frac{\hbar}{2i}\left(\psi^*\dot{\psi} - \psi\dot{\psi}^*\right)\right]$$

$$\partial_t \frac{\partial L_{Density}}{\partial(\partial_t \psi^*)} = -\partial_t \frac{\hbar}{2i}\psi = -\frac{\partial}{\partial t}\frac{\hbar}{2i}\psi$$

Putting them all together:

$$V\psi + \frac{\hbar}{2i}\dot{\psi} - \frac{\hbar^2}{2m}\partial_x^2\psi - \frac{\hbar^2}{2m}\partial_y^2\psi - \frac{\hbar^2}{2m}\partial_z^2\psi + \frac{\partial}{\partial t}\frac{\hbar}{2i}\psi = 0$$

With some rearranging it becomes the more familiar form of Schrodinger's Wave Equation for a particle in an unspecified potential

$$\frac{\hbar^2}{2m}\nabla^2\psi - V\psi = -i\hbar\frac{\partial}{\partial t}\psi$$

The conjugate momenta are found from:

$$p_\psi = \frac{\partial L_{Density}}{\partial \dot{\psi}}$$

$$p_\psi = \frac{\partial L_{Density}}{\partial \dot{\psi}}\left[\frac{\hbar^2}{2m}\left(\partial_x\psi\partial_x\psi^* + \partial_y\psi\partial_y\psi^* + \partial_z\psi\partial_z\psi^*\right) + V\psi^*\psi + \frac{\hbar}{2i}\left(\psi^*\dot{\psi} - \psi\dot{\psi}^*\right)\right]$$

$$p_\psi = \frac{\hbar}{2i}(\psi^*)$$

and

$$p_{\psi^*} = \frac{\partial L_{Density}}{\partial \dot{\psi}^*}$$

$$p_{\psi^*} = \frac{\partial L_{Density}}{\partial \dot{\psi}^*}\left[\frac{\hbar^2}{2m}\left(\partial_x\psi\partial_x\psi^* + \partial_y\psi\partial_y\psi^* + \partial_z\psi\partial_z\psi^*\right) + V\psi^*\psi + \frac{\hbar}{2i}\left(\psi^*\dot{\psi} - \psi\dot{\psi}^*\right)\right]$$

$$p_{\psi^*} = \frac{\hbar}{2i}\psi$$

4. First show that the following transformation is canonical:

$$x=\frac{1}{\alpha}\left(\sqrt{2P_1}\sin Q_1+P_2\right),\ p_x=\frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1-Q_2\right)$$
$$y=\frac{1}{\alpha}\left(\sqrt{2P_1}\cos Q_1+Q_2\right),\ p_x=-\frac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1-P_2\right)$$

A transformation is canonical if the following is an exact differential: I found this really cool relationship but it turned out to be not worth anything… it looks like angular momentum

$$x=\frac{1}{\alpha}\left(\sqrt{2P_1}\sin Q_1+P_2\right),\ p_x=\frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1-Q_2\right)$$
$$y=\frac{1}{\alpha}\left(\sqrt{2P_1}\cos Q_1+Q_2\right),\ p_y=-\frac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1-P_2\right)$$
$$xp_y=-\frac{\alpha}{2}\frac{1}{\alpha}\left(\sqrt{2P_1}\sin Q_1+P_2\right)\left(\sqrt{2P_1}\sin Q_1-P_2\right)=-\frac{1}{2}\left(2P_1\sin^2 Q_1-P_2^2\right)$$
$$yp_x=\frac{\alpha}{2}\frac{1}{\alpha}\left(\sqrt{2P_1}\cos Q_1+Q_2\right)\left(\sqrt{2P_1}\cos Q_1-Q_2\right)=\frac{1}{2}\left(2P_1\cos^2 Q_1-Q_2^2\right)$$
$$yp_x-xp_y=\frac{1}{2}\left(2P_1\cos^2 Q_1-Q_2^2\right)+\frac{1}{2}\left(2P_1\sin^2 Q_1-P_2^2\right)$$
$$yp_x-xp_y=\frac{1}{2}\left(2P_1-Q_2^2-P_2^2\right)$$

Back to proving it is an exact differential… the first part is to express $p_x dx$ pxdx in terms of P's and Q's

$$x=\frac{1}{\alpha}\left(\sqrt{2P_1}\sin Q_1+P_2\right)$$
$$dx=\frac{1}{\alpha}\left(\frac{1}{\sqrt{2P_1}}\sin Q_1 dP_1+\sqrt{2P_1}\cos Q_1 dQ_1+dP_2\right)$$
$$p_x=\frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1-Q_2\right)$$
$$p_x dx=\frac{1}{\alpha}\frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1-Q_2\right)\left(\frac{1}{\sqrt{2P_1}}\sin Q_1 dP_1+\sqrt{2P_1}\cos Q_1 dQ_1+dP_2\right)$$
$$2p_x dx=\sqrt{2P_1}\cos Q_1\frac{1}{\sqrt{2P_1}}\sin Q_1 dP_1+\sqrt{2P_1}\cos Q_1\sqrt{2P_1}\cos Q_1 dQ_1+\sqrt{2P_1}\cos Q_1 dP_2$$
$$-Q_2\frac{1}{\sqrt{2P_1}}\sin Q_1 dP-Q_2\sqrt{2P_1}\cos Q_1 dQ_1-Q_2 dP_2$$
$$2p_x dx=\cos Q_1\sin Q_1 dP_1+2P_1\cos^2 Q_1 dQ_1+\sqrt{2P_1}\cos Q_1 dP_2$$
$$-Q_2\frac{1}{\sqrt{2P_1}}\sin Q_1 dP-Q_2\sqrt{2P_1}\cos Q_1 dQ_1-Q_2 dP_2$$

Now express $p_y dy$ in terms of P's and Q's

$$y = \frac{1}{\alpha}\left(\sqrt{2P_1}\cos Q_1 + Q_2\right)$$

$$dy = \frac{1}{\alpha}\left(\frac{1}{\sqrt{2P_1}}\cos Q_1 dP - \sqrt{2P_1}\sin Q_1 dQ_1 + dQ_2\right)$$

$$p_y = -\frac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right)$$

$$p_y dy = -\frac{1}{\alpha}\frac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right)\left(\frac{1}{\sqrt{2P_1}}\cos Q_1 dP - \sqrt{2P_1}\sin Q_1 dQ_1 + dQ_2\right)$$

$$-2p_y dy = \sqrt{2P_1}\sin Q_1 \frac{1}{\sqrt{2P_1}}\cos Q_1 dP - \sqrt{2P_1}\sin Q_1\sqrt{2P_1}\sin Q_1 dQ_1 + \sqrt{2P_1}\sin Q_1 dQ_2$$

$$-P_2\frac{1}{\sqrt{2P_1}}\cos Q_1 dP + P_2\sqrt{2P_1}\sin Q_1 dQ_1 - P_2 dQ_2$$

$$-2p_y dy = \sin Q_1\cos Q_1 dP - 2P_1\sin^2 Q_1 dQ_1 + \sqrt{2P_1}\sin Q_1 dQ_2 - P_2\frac{1}{\sqrt{2P_1}}\cos Q_1 dP + P_2\sqrt{2P_1}\sin Q_1 dQ_1 - P_2 dQ_2$$

Now get an expression for $p_x dx + p_y dy$ in terms of P's and Q's:

$$\begin{aligned}2p_x dx + 2p_y dy = {} & \cos Q_1\sin Q_1 dP_1 + 2P_1\cos^2 Q_1 dQ_1 + \sqrt{2P_1}\cos Q_1 dP_2 \\ & - Q_2\frac{1}{\sqrt{2P_1}}\sin Q_1 dP - Q_2\sqrt{2P_1}\cos Q_1 dQ_1 - Q_2 dP_2 \\ & - \sin Q_1\cos Q_1 dP + 2P_1\sin^2 Q_1 dQ_1 - \sqrt{2P_1}\sin Q_1 dQ_2 \\ & + P_2\frac{1}{\sqrt{2P_1}}\cos Q_1 dP - P_2\sqrt{2P_1}\sin Q_1 dQ_1 + P_2 dQ_2\end{aligned}$$

$$\begin{aligned}2p_x dx + 2p_y dy = {} & 2P_1 dQ_1 + P_2 dQ_2 - Q_2 dP_2 \\ & + \sqrt{2P_1}\cos Q_1 dP_2 + P_2\frac{1}{\sqrt{2P_1}}\cos Q_1 dP_1 - P_2\sqrt{2P_1}\sin Q_1 dQ_1 \\ & - Q_2\frac{1}{\sqrt{2P_1}}\sin Q_1 dP_1 - Q_2\sqrt{2P_1}\cos Q_1 dQ_1 - \sqrt{2P_1}\sin Q_1 dQ_2\end{aligned}$$

$$2p_x dx + 2p_y dy = 2P_1 dQ_1 + P_2 dQ_2 - Q_2 dP_2 + d\left(P_2\sqrt{2P_1}\cos Q_1 - Q_2\sqrt{2P_1}\sin Q_1\right)$$

$$p_x dx + p_y dy = P_1 dQ_1 + P_2 dQ_2 - \tfrac{1}{2}P_2 dQ_2 - \tfrac{1}{2}Q_2 dP_2 + d\,\tfrac{1}{2}\left(P_2\sqrt{2P_1}\cos Q_1 - Q_2\sqrt{2P_1}\sin Q_1\right)$$

$$p_x dx + p_y dy = P_1 dQ_1 + P_2 dQ_2 + d\,\tfrac{1}{2}\left(P_2\sqrt{2P_1}\cos Q_1 - Q_2\sqrt{2P_1}\sin Q_1 - P_2 Q_2\right)$$

The final relationship is what is needed to show the exact differential:

$$p_x dx + p_y dy = P_1 dQ_1 + P_2 dQ_2 + d\,\tfrac{1}{2}\left(P_2\sqrt{2P_1}\cos Q_1 - Q_2\sqrt{2P_1}\sin Q_1 - P_2 Q_2\right)$$

So this is the final part of the proof to show that it is an exact differential:

$$\begin{aligned}
&dF \stackrel{?}{=} \sum_i p_i dq_i - \sum_j P_j dQ_j \\
&\sum_i p_i dq_i - \sum_j P_j dQ_j = \\
&p_x dx + p_y dy - P_1 dQ_1 - P_2 dQ_2 = \\
&P_1 dQ_1 + P_2 dQ_2 + d\,\tfrac{1}{2}\left(P_2\sqrt{2P_1}\cos Q_1 - Q_2\sqrt{2P_1}\sin Q_1 - P_2 Q_2\right) - P_1 dQ_1 - P_2 dQ_2 = \\
&d\,\tfrac{1}{2}\left(P_2\sqrt{2P_1}\cos Q_1 - Q_2\sqrt{2P_1}\sin Q_1 - P_2 Q_2\right) = \\
&dF \\
&where \\
&F = \tfrac{1}{2}\left(P_2\sqrt{2P_1}\cos Q_1 - Q_2\sqrt{2P_1}\sin Q_1 - P_2 Q_2\right)
\end{aligned}$$

After construction the Hamiltonian for a particle of charge q moving in a plane that is perpendicular to a constant magnetic field B, use the transformation to express the Hamiltonian in the (Q,P) and obtain the motion of the particle as a function of time.

OK so the next thing to do is construct the Hamiltonian. First express the kinetic energy in x and y (assuming B is parallel to the z axis):

$$T = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right)$$

Now the potential energy due to the Magnetic field:

$$\vec{B} = B\hat{z}$$

$$V = -\frac{q}{c}\vec{v}\cdot\vec{A} = -\frac{q}{2c}\vec{v}\cdot\left(\vec{B}\times\vec{R}\right) = -\frac{q}{2c}\vec{v}\cdot\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ 0 & 0 & B\\ x & y & z\end{vmatrix}$$

$$V = -\frac{Bq}{2c}\left(\dot{x}\hat{x} + \dot{y}\hat{y}\right)\cdot\left(-y\hat{x} + x\hat{y}\right) = -\frac{Bq}{2c}\left(-y\dot{x} + x\dot{y}\right)$$

The Lagrangian is:

$$L = T - V = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{Bq}{2c}\left(x\dot{y} - y\dot{x}\right)$$

The conjugate momenta are:

$$L = T - V = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{Bq}{2c}\left(x\dot{y} - y\dot{x}\right)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left\{\tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{Bq}{2c}\left(x\dot{y} - y\dot{x}\right)\right\}$$

$$p_x = m\dot{x} - y\frac{Bq}{2c} = m\dot{x} - \frac{\alpha^2}{2} y$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}}\left\{\tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{Bq}{2c}\left(x\dot{y} - y\dot{x}\right)\right\}$$

$$p_y = m\dot{y} + x\frac{Bq}{2c} = m\dot{y} + \frac{\alpha^2}{2} x$$

Or you can find the momenta from the T-1 method:

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{Bq}{2c}\left(x\dot{y} - y\dot{x}\right) = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{\alpha^2}{2}\left(x\dot{y} - y\dot{x}\right)$$

$$L = \tfrac{1}{2}\begin{bmatrix}\dot{x} & \dot{y}\end{bmatrix}\begin{vmatrix} m & 0 \\ 0 & m \end{vmatrix}\begin{bmatrix}\dot{x} \\ \dot{y}\end{bmatrix} + \begin{bmatrix}\dot{x} & \dot{y}\end{bmatrix}\begin{vmatrix} -\frac{\alpha^2 y}{2} & 0 \\ 0 & \frac{\alpha^2 x}{2} \end{vmatrix}\begin{bmatrix}\dot{x} \\ \dot{y}\end{bmatrix} = \tfrac{1}{2}\tilde{\dot{\eta}} T \dot{\eta} + \tilde{\dot{\eta}} a + L_0 \quad \text{(8.23 form)}$$

$$p = T\dot{\eta} + a = \begin{vmatrix} m & 0 \\ 0 & m \end{vmatrix}\begin{bmatrix}\dot{x} \\ \dot{y}\end{bmatrix} + \begin{vmatrix} -\frac{\alpha^2 y}{2} & 0 \\ 0 & \frac{\alpha^2 x}{2} \end{vmatrix}$$

$$p_x = m\dot{x} - \frac{\alpha^2 y}{2}$$

$$p_y = m\dot{y} + \frac{\alpha^2 x}{2}$$

Find the T-1 matrix:

$$T = \begin{vmatrix} m & 0 \\ 0 & m \end{vmatrix}$$

$$T^{-1} = \frac{1}{m^2}\begin{vmatrix} m & 0 \\ 0 & m \end{vmatrix} = \begin{vmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{vmatrix}$$

Form via equation 8.27

$$H = \tfrac{1}{2}\left(\tilde{p} - \tilde{a}\right)T^{-1}(p-a) - L_0$$

$$H = \tfrac{1}{2}\begin{vmatrix} m & 0 \\ 0 & m \end{vmatrix}\begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix}\begin{vmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{vmatrix}\begin{vmatrix} m & 0 \\ 0 & m \end{vmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

$$H = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$H = \frac{1}{2m}\left(p_x + \frac{\alpha^2 y}{2}\right)^2 + \frac{1}{2m}\left(p_y - \frac{\alpha^2 x}{2}\right)^2$$

The Hamiltonian is the same as in equation 8.34

$$H = \frac{1}{2m}\left(p_x + \frac{\alpha^2 y}{2}\right)^2 + \frac{1}{2m}\left(p_y - \frac{\alpha^2 x}{2}\right)^2$$

Convert to new coordinates:

$$x = \frac{1}{\alpha}\left(\sqrt{2P_1}\sin Q_1 + P_2\right),\ p_x = \frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1 - Q_2\right)$$

$$y = \frac{1}{\alpha}\left(\sqrt{2P_1}\cos Q_1 + Q_2\right),\ p_x = -\frac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right)$$

$$H = \frac{1}{2m}\left(\frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1 - Q_2\right) + \frac{\alpha^2}{2}\frac{1}{\alpha}\left(\sqrt{2P_1}\cos Q_1 + Q_2\right)\right)^2$$

$$+\frac{1}{2m}\left(-\frac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right) - \frac{\alpha^2}{2}\frac{1}{\alpha}\left(\sqrt{2P_1}\sin Q_1 + P_2\right)\right)^2$$

$$H = \frac{1}{2m}\left(\frac{\alpha}{2}\right)^2\left(\sqrt{2P_1}\cos Q_1 - Q_2 + \sqrt{2P_1}\cos Q_1 + Q_2\right)^2$$

$$+\frac{1}{2m}\left(\frac{\alpha}{2}\right)^2\left(\sqrt{2P_1}\sin Q_1 - P_2 + \sqrt{2P_1}\sin Q_1 + P_2\right)^2$$

$$H = \frac{1}{2m}\left(\frac{\alpha}{2}\right)^2\left(2\sqrt{2P_1}\cos Q_1\right)^2 + \frac{1}{2m}\left(\frac{\alpha}{2}\right)^2\left(2\sqrt{2P_1}\sin Q_1\right)^2$$

$$H = \frac{1}{2m}\left(\frac{\alpha}{2}\right)^2 8P_1$$

$$H = \frac{\alpha^2}{m}P_1$$

$$H = \frac{\alpha^2}{m} P_1 = E$$
$$\dot{Q}_1 = \frac{\partial H}{\partial P_1} = \frac{\alpha^2}{m}$$
$$Q_1 = \frac{\alpha^2}{m} t + \beta$$
$$P_1 = \frac{m}{\alpha^2} E$$
$$x = \frac{1}{\alpha}\left(\sqrt{2\frac{m}{\alpha^2} E} \sin\left(\frac{\alpha^2}{m} t + \beta \right) + P_2 \right)$$
$$y = \frac{1}{\alpha}\left(\sqrt{2\frac{m}{\alpha^2} E} \cos\left(\frac{\alpha^2}{m} t + \beta \right) + Q_2 \right)$$
$$LET : @\, t = -\frac{m\beta}{\alpha^2}, x = 0, y = 0$$
$$0 = \frac{1}{\alpha}\left(\sqrt{2\frac{m}{\alpha^2} E} \sin(0) + P_2 \right) \to P_2 = 0$$
$$0 = \frac{1}{\alpha}\left(\sqrt{2\frac{m}{\alpha^2} E} + Q_2 \right) \to Q_2 = -\sqrt{2\frac{m}{\alpha^2} E}$$
$$x = \frac{1}{\alpha}\left(\sqrt{2\frac{m}{\alpha^2} E} \sin\left(\frac{\alpha^2}{m} t + \beta \right) \right)$$
$$y = \frac{1}{\alpha}\left(\sqrt{2\frac{m}{\alpha^2} E} \cos\left(\frac{\alpha^2}{m} t + \beta \right) - \sqrt{2\frac{m}{\alpha^2} E} \right)$$

This is the equation of a circle.

$$\omega = \frac{\alpha^2}{m} = \frac{qB}{mc}$$
$$x = \left(\sqrt{\frac{2mE}{\omega^2}} \sin(\omega t + \beta) \right)$$
$$y - \sqrt{\frac{2mE}{\omega^2}} = \left(\sqrt{\frac{2mE}{\omega^2}} \cos(\omega t + \beta) \right)$$
$$r^2 = (x - x_0)^2 + (y - y_0)^2 = \frac{2mE}{\omega^2}$$

Note: the angular momentum is constant also: $yp_x - xp_y = \frac{1}{2}\left(2P_1 - Q_2^2 - P_2^2\right) = \frac{E}{\omega} - 2\frac{E}{\omega} - 0 = -\frac{E}{\omega}$