Quantum Mechanics — Meg Noel (9/10)

18.4.1 By taking the divergence of Eq. 18.4.5 show that the continuity equation must be obeyed if Maxwell's equations are to be mutually consistent.

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}$$

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{B} - \vec{\nabla}\cdot\left(\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right) = \frac{4\pi}{c}\vec{\nabla}\cdot\vec{j}$$ divergence of curl is zero

$$\vec{\nabla}\cdot\vec{j} = -\vec{\nabla}\cdot\frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t} = -\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\frac{\vec{E}}{4\pi}\right) = -\frac{\partial\rho}{\partial t}$$

$\vec{\nabla}\cdot\vec{E} = \rho\, 4\pi$ Gaussian units Gauss Law

$$\vec{\nabla}\vec{j} + \frac{\partial\rho}{\partial t} = 0$$

18.4.2 Calculate $\vec{E}$ and $\vec{B}$ ~~to~~ corresponding to $(\vec{A},\phi)$ and $(\vec{A}',\phi')$ using equations 18.4.7 and 18.4.9 and verify the above claim.

curl of gradient is zero

$$\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times(\vec{A}-\vec{\nabla}\Lambda) = \vec{\nabla}\times\vec{A} - \vec{\nabla}\times\vec{\nabla}\Lambda = \vec{\nabla}\times\vec{A} = \vec{B}$$

$$\therefore\ \vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\vec{A}'$$

~~$\vec{E}' = -\frac{1}{c}\frac{\partial\vec{A}'}{\partial t} - \nabla\phi' = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{A}-\vec{\nabla}\Lambda) - \vec{\nabla}\left(\phi + \frac{1}{c}\frac{\partial\Lambda}{\partial t}\right)$~~

~~$\vec{\nabla}\times\vec{E}' = \vec{\nabla}\times\left(-\frac{1}{c}\frac{\partial}{\partial t}\vec{A}\right) + \vec{\nabla}\times\vec{\nabla}\Lambda - \vec{\nabla}\times\vec{\nabla}\left(\phi + \frac{1}{c}\frac{\partial\Lambda}{\partial t}\right) =$~~ $\vec{\nabla}\times\left(-\frac{1}{c}\frac{\partial}{\partial t}\vec{A}\right) = \vec{\nabla}\times\vec{E}$

~~$\vec{E}' = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\Lambda - \vec{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\Lambda$~~

~~$-\vec{\nabla}\phi = \vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ 18.4.9~~

~~$\Rightarrow \vec{E}' = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\Lambda + \vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\Lambda = \vec{E} - \vec{\nabla}\left(\Lambda + \frac{1}{c}\frac{\partial\Lambda}{\partial t}\right)$~~

$$\vec{A}' = \vec{A} - \vec{\nabla}\Lambda$$
$$\phi' = \phi + \frac{1}{c}\frac{\partial\Lambda}{\partial t}$$

~~$\phi' - \vec{A}' = \phi - \vec{A} + \left(\frac{1}{c}\frac{\partial}{\partial t}\Lambda + \vec{\nabla}\Lambda\right)$~~

$$\frac{\partial\vec{A}}{\partial t} = \frac{\partial\vec{A}'}{\partial t} + \vec{\nabla}\frac{\partial\Lambda}{\partial t} = \frac{\partial\vec{A}'}{\partial t} + \vec{\nabla}(c\phi' - c\phi)$$

$$\Rightarrow \frac{\partial\vec{A}}{\partial t} + \vec{\nabla}c\phi = \frac{\partial\vec{A}'}{\partial t} + \vec{\nabla}c\phi' = -c\vec{E} \Rightarrow \vec{E} = -\left(\frac{\partial\vec{A}}{\partial t} + \vec{\nabla}\phi\right) = -\left(\frac{\partial\vec{A}}{\partial t} + \vec{\nabla}\phi'\right)$$

18.4.3 Suppose we are given some $\vec{A}$ and ϕ that do not obey the Coulomb gauge conditions. Let us see how they can be transformed to the Coulomb gauge.

(1) Show that if we choose $\Lambda(\vec{r},t) = -c\int_{-\infty}^{t} \phi(\vec{r},t')dt'$

and tranform to $(\vec{A}', \phi')$ then $\phi'=0$. $\vec{A}'$ is just $\vec{A} - \vec{\nabla}\Lambda$, with $\vec{\nabla}\cdot\vec{A}'$ not necessarily zero.

$$\Lambda(\vec{r},t) = -c\int_{-\infty}^{t}\phi(\vec{r},t')dt'$$

$$\frac{\partial\Lambda}{\partial t} = -c\phi(\vec{r},t') \Rightarrow \phi = \frac{-1}{c}\frac{\partial\Lambda}{\partial t} \Rightarrow \phi' = \phi + \frac{1}{c}\frac{\partial\Lambda}{\partial t} = \frac{-1}{c}\frac{\partial\Lambda}{\partial t} + \frac{1}{c}\frac{\partial\Lambda}{\partial t} = 0$$

(2) Show that if we gauge transform one more to $(\vec{A}'', \phi'')$ via $\Lambda' = \frac{-1}{4\pi}\int\frac{\vec{\nabla}\cdot\vec{A}'(\vec{r}',t)d^3r'}{|\vec{r}-\vec{r}'|}$ then $\vec{\nabla}\cdot\vec{A}''=0$

because $\nabla^2\frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\delta^3(\vec{r}-\vec{r}')$

$$\vec{A}' = \vec{A} - \nabla\Lambda \Rightarrow \vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} - \nabla^2\Lambda$$

$$\vec{A}'' = \vec{A}' - \nabla\Lambda' = \vec{A} - \vec{\nabla}\Lambda - \vec{\nabla}\left(\frac{-1}{4\pi}\int\frac{\vec{\nabla}\cdot\vec{A}'(\vec{r}',t)d^3\vec{r}'}{|\vec{r}-\vec{r}'|}\right) = \vec{A} - \vec{\nabla}\Lambda + \vec{A}'$$

$$\vec{\nabla}\cdot\vec{A}'' = \vec{\nabla}\cdot\vec{A} - \nabla^2\Lambda + (\vec{\nabla}\cdot\vec{A}') = \vec{\nabla}\cdot\vec{A} - \nabla^2\Lambda - \vec{\nabla}\cdot\vec{A}' + \nabla^2\Lambda = 0$$

(3) Verify that ϕ'' is also zero by using $\vec{\nabla}\cdot\vec{E}=0$.

$$\phi'' = \phi' + \frac{1}{c}\frac{\partial\Lambda'}{\partial t} = \phi + \frac{1}{c}\frac{\partial\Lambda}{\partial t} + \frac{1}{c}\frac{\partial\Lambda'}{\partial t} = \phi + \frac{1}{c}\frac{\partial\Lambda}{\partial t} + \frac{1}{c}\frac{\partial}{\partial t}\left[-\frac{1}{4\pi}\int\frac{\vec{\nabla}\cdot\vec{A}d^3\vec{r}}{|\vec{r}-\vec{r}'|}\right]$$

$$\phi'' = \phi + \frac{1}{c}\frac{\partial\Lambda}{\partial t} - \frac{1}{4\pi}\int\frac{\vec{\nabla}\cdot\frac{\partial\vec{A}}{c\partial t}d^3\vec{r}}{|\vec{r}-\vec{r}'|} = \phi + \frac{1}{c}\frac{\partial\Lambda}{\partial t} + \frac{1}{4\pi}\int\frac{\vec{\nabla}\cdot(\vec{E}+\nabla\phi')d^3\vec{r}}{|\vec{r}-\vec{r}'|}$$

by 18.4.9 $\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \nabla\phi$

$$\phi'' = \phi + \frac{1}{c}\frac{\partial\Lambda}{\partial t} + \frac{1}{4\pi}\int\frac{\nabla^2\phi' d^3\vec{r}}{|\vec{r}-\vec{r}'|} = \phi + \frac{1}{c}\frac{\partial\Lambda}{\partial t} - \phi' = 0$$

because $\phi' = \phi + \frac{1}{c}\frac{\partial\Lambda}{\partial t}$

(4) Show that if we want to make any further gauge transformations within the Coulomb gauge, Λ must be time independent and obey $\nabla^2\Lambda=0$. If we demand that $|\vec{A}|\to 0$ at spatial infinity, the $\vec{A}$ becomes unique.

if $|\vec{A}| \to 0$ then $\vec{\nabla}\cdot\vec{A}' \to \vec{\nabla}\cdot\vec{A}$

from $\vec{A}' = \vec{A} - \nabla\Lambda \to \vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} - \nabla^2\Lambda$

true iff $\nabla^2\Lambda = 0$

18.4.4 Proof of Gauge invariance in the Schrödinger Approach.

(1) Write $\hat{H}$ for a particle in the potentials $(\vec{A}, \phi)$

Canonical momentum of a particle: $\vec{p} = m\vec{v} + \frac{q\vec{A}}{c}$ (2.27) q = charge, c = speed of light in vac.

The potential of a point particle in $(\vec{A}, \phi)$: $V = q\phi$

$$\hat{H} = \frac{[\vec{p} - q\vec{A}/c]^2}{2m} + q\phi \qquad (2.6.2)$$

(2) Write down $\hat{H}_\Lambda$, the Hamiltonian obtained by gauge transforming the potentials.

$\vec{A}' = \vec{A} - \nabla\Lambda$ (18.4.12) $\quad \phi' = \phi + \frac{1}{c}\frac{\partial \Lambda}{\partial t}$ (18.4.13)

$$\hat{H}'_\Lambda = \frac{[\vec{p} - \frac{q}{c}(\vec{A} - \nabla\Lambda)]^2}{2m} + q\left[\phi + \frac{1}{c}\frac{\partial\Lambda}{\partial t}\right] = \frac{(\vec{p} - \frac{q}{c}\vec{A} + \frac{q}{c}\nabla\Lambda)^2}{2m} + q\phi + \frac{q}{c}\frac{\partial\Lambda}{\partial t}$$

(3) Show that if $\Psi(\vec{r}, t)$ is a solution to Schrödinger's equation with the Hamiltonian H, then $\Psi_\Lambda(\vec{r}, t)$ given in Equation 18.4.33 is the corresponding solution with $H \to H_\Lambda$

$\Psi_\Lambda = e^{-iq\Lambda(\vec{r},t)/\hbar c}\,\Psi$ (18.4.33) better derivation $\Rightarrow$

$$H'\Psi_\Lambda = H' e^{-iq\Lambda/\hbar c}\Psi = \frac{1}{2m}[(\vec{p} - \frac{q}{c}\vec{A}) + \frac{q}{c}\nabla\Lambda]^2 e^{iq\Lambda/\hbar c}\Psi + q\phi e^{iq\Lambda/\hbar c}\Psi + \frac{q}{c}\frac{\partial\Lambda}{\partial t}(e^{iq\Lambda/\hbar c}\Psi)$$

~~$H'\Psi_\Lambda = \frac{1}{2m}[\vec{p} - \frac{q}{c}\vec{A} + \frac{q}{c}\nabla\Lambda][(\vec{p} - \frac{q}{c}\vec{A})e^{iq\Lambda/\hbar c}\Psi + \frac{q}{c}e^{iq\Lambda/\hbar c}\nabla\Lambda\Psi + \frac{q}{c}\frac{i}{\hbar c}(\nabla\Lambda)e^{\dots}]$~~

$$= \frac{1}{2m}(\vec{p} - \frac{q}{c}\vec{A})^2\Psi e^{iq\Lambda/\hbar c} + \left\{\frac{1}{2m}(\vec{p} - \frac{q}{c}\vec{A})(\frac{q}{c}\nabla\Lambda) + \frac{q}{c}\nabla\Lambda\frac{1}{2m}(p - \frac{q}{c}\vec{A}) + \frac{q^2}{c}\nabla^2\Lambda\right\}\Psi e^{iq\Lambda/\hbar c}$$

$$+ q\phi e^{iq\Lambda/\hbar c}\Psi + \frac{q}{c}\frac{\partial\Lambda}{\partial t}(e^{iq\Lambda/\hbar c}\Psi)$$

$\vec{p} = -i\hbar\nabla$ and $\vec{\nabla}\cdot(\vec{A}f) = (\vec{\nabla}\cdot\vec{A})f + \vec{A}\cdot(\vec{\nabla}f)$

$$[\vec{p} - \frac{q}{c}\vec{A} + \frac{q}{c}\nabla\Lambda]^2 = [-i\hbar\nabla - \frac{qA}{c} + \frac{q}{c}\nabla\Lambda]\cdot[i\hbar\nabla - \frac{qA}{c} + \frac{q}{c}\nabla\Lambda]$$

$$-\hbar^2\nabla^2 + i\hbar\nabla\cdot(\frac{q\vec{A}}{c}) - i\hbar\frac{q}{c}\nabla\Lambda - \frac{q\vec{A}}{c}(i\hbar\nabla) + \frac{q^2A^2}{c} - \frac{q^2\vec{A}}{c^2}\cdot\nabla\Lambda$$

$$+ \frac{i\hbar q}{c}\nabla\Lambda\nabla - \frac{q^2}{c^2}(\nabla\Lambda)\cdot\vec{A} + \frac{q^2}{c^2}\nabla^2\Lambda$$

$$= -\hbar^2\nabla^2 + i\hbar\nabla\cdot(\frac{q\vec{A}}{c}) - \frac{q\vec{A}}{c}\cdot i\hbar\nabla + \frac{q^2A^2}{c} - \frac{q^2}{c^2}[\vec{A}\cdot\nabla\Lambda + \nabla\Lambda\cdot\vec{A}] = [\vec{p} - \frac{q}{c}\vec{A}]^2$$

$$\Rightarrow H'\Psi_\Lambda = \frac{1}{2m}(\vec{p} - \frac{q}{c}\vec{A})^2\Psi e^{iq\Lambda/\hbar c} + q\phi e^{iq\Lambda/\hbar c}\Psi + \frac{q}{c}\frac{\partial\Lambda}{\partial t}(e^{iq\Lambda/\hbar c}\Psi)$$

$$i\hbar\frac{\partial}{\partial t}\Psi_\Lambda = i\hbar\frac{\partial}{\partial t}(e^{-iq\Lambda/\hbar c}\Psi) = \left[i\hbar\frac{\partial\Psi}{\partial t} + q\frac{\partial\Lambda}{\partial t}\left(\frac{1}{c}\Psi\right)\right]e^{-iq\Lambda/\hbar c}$$ but $i\hbar\frac{\partial\Psi}{\partial t} = H\Psi \Rightarrow$

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi = \left[\frac{(\vec{p} - \frac{q}{c}\vec{A})^2}{2m} + q\phi\right]\Psi$$

$$\therefore \quad i\hbar \frac{\partial}{\partial t}\Psi_\Lambda = \left[\frac{(\vec{p} - \frac{q}{c}\vec{A})^2}{2m} + e\phi\right]\Psi e^{-iq\Lambda/\hbar c} + \frac{q}{c}\frac{\partial \Lambda}{\partial t}\Psi e^{-iq\Lambda/\hbar c}$$

$$\therefore \quad i\hbar\frac{\partial}{\partial t}\Psi e^{-iq\Lambda/\hbar c} = H'\Psi e^{-iq\Lambda/\hbar c}$$

(neater) better derivation in Schaum's $\Longrightarrow$

18.5.1 (1) By going through the derivation, argue that we can take $e^{i\vec{k}\cdot\vec{r}}$ factor into account exactly, by replacing $\vec{p}_f$ by $\vec{p}_f - \hbar\vec{k}$ in Eq. 18-5-19.

$$e^{-i\vec{p}_f\cdot\vec{r}/\hbar} \rightarrow e^{-i(\vec{p}_f - \hbar\vec{k})\cdot\vec{r}/\hbar} = e^{-i\vec{p}_f\cdot\vec{r}/\hbar}\, e^{+i\vec{k}\cdot\vec{r}}$$

so the factor of $e^{+\vec{k}\cdot\vec{r}}$ is taken into account exactly.

(2) Verify the claim made above about the electron momentum distribution.

$$\int_0^\infty \int_{-1}^{+1} \int_0^{2\pi} e^{-p_f r\cos\theta/\hbar}\, e^{i\vec{k}\cdot\vec{r}}\, e^{-r/a_0}\, r^2\, dr\, d\cos\theta\, d\phi$$

$$\vec{k}\cdot\vec{r} = (k_x\hat{x} + k_y\hat{y} + k_z\hat{z})\cdot(r\cos\phi\sin\theta\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z})$$

$$\vec{k}\cdot\vec{r} = k_x r\cos\phi\sin\theta + k_y r\sin\theta\sin\phi + k_z r\cos\theta$$

$$= \int_0^\infty \int_{-1}^{+1} e^{-p_f r\cos\theta/\hbar}\, r^2\, dr\, d\cos\theta \int_0^{2\pi} \left(e^{ik_x r\cos\phi\sin\theta}\right)\left(e^{ik_y r\sin\theta\sin\phi}\right)\left(e^{ik_z r\cos\theta}\right) d\phi$$

NOTE BOOK

18.44(3) Schaum's

$$\hat{H} = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)\cdot\left(\vec{p} - \frac{q}{c}\vec{A}\right) + q\phi$$

$$\hat{H}' = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A} - \frac{q}{c}\nabla A\right)\cdot\left(\vec{p} - \frac{q}{c}\vec{A} - \frac{q}{c}\nabla A\right) + q\phi - \frac{q}{c}\frac{\partial A}{\partial t}$$

$$\hat{H}|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$$

$$\left[\frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)^2 + q\phi\right]\psi(\vec{r},t) = i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}$$

and

$$\hat{H}'|\psi'\rangle = i\hbar\frac{\partial}{\partial t}|\psi'\rangle$$

$$\left[\frac{1}{2m}\left(-i\hbar\nabla - \frac{q}{c} - \frac{q}{c}\nabla A\right)^2 + \frac{q}{c} - \frac{q}{c}\frac{\partial A}{\partial t}\right]\psi'(\vec{r},t) = i\hbar\frac{\partial\psi'}{\partial t}(\vec{r},t)$$

Let $\psi'(\vec{r},t) = e^{iq\Lambda/\hbar c}\,\psi(\vec{r},t)$

$$i\hbar\frac{\partial\psi'}{\partial t} = \frac{q}{c}\frac{\partial\Lambda}{\partial t}e^{iq\Lambda/\hbar c}\psi + e^{iq\Lambda/\hbar c}\,i\hbar\left(\frac{\partial\psi}{\partial t}\right)$$

$$= -\frac{q}{c}\frac{\partial\Lambda}{\partial t}\psi' + e^{iq\Lambda/\hbar c}\left[\frac{1}{2m}\left(-i\hbar\nabla - \frac{q}{c}\vec{A}\right)^2 + q\phi\right]e^{iq\Lambda/\hbar c}\psi'$$

$$\Rightarrow i\hbar\frac{\partial\psi'}{\partial t} = \left[-\frac{q}{c}\frac{\partial\Lambda}{\partial t} + q\phi\right]\psi' + e^{iq\Lambda/\hbar c}\left[\frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)^2\right]e^{-iq\Lambda/\hbar c}\psi'$$

$$= \left[-\frac{q}{c}\frac{\partial\Lambda}{\partial t} + q\phi\right]\psi' + \left(-i\hbar\nabla - \frac{q}{c}\vec{A}\right)\cdot\left[e^{iq\Lambda/\hbar c}\left(-\frac{q}{c}\nabla\Lambda - i\hbar\nabla - \frac{q}{c}A\right)\right]\psi'$$

$$= \left[-\frac{q}{c}\frac{\partial\Lambda}{\partial t} + q\phi\right]\psi' + e^{-iq\Lambda/\hbar c}\left(-\frac{q}{c}\nabla\Lambda - i\hbar\nabla - \frac{q}{c}\vec{A}\right)\cdot\left(-\frac{q}{c}\nabla\Lambda - i\hbar\nabla - \frac{q}{c}\vec{A}\right)\psi'$$

$$= \left[-\frac{q}{c}\frac{\partial\Lambda}{\partial t} + q\phi + \frac{1}{2m}\left(-\frac{q}{c}\vec{\nabla}\Lambda - i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)^2\right]\psi'$$

$\therefore$ $\psi' = e^{iq\Lambda/\hbar c}\psi$ satisfies

$$H|\psi\rangle = -i\hbar\frac{\partial}{\partial t}|\psi\rangle \quad \text{and} \quad H'|\psi'\rangle = -i\hbar\frac{\partial}{\partial t}|\psi'\rangle$$

NOTE BOOK

18.5.1 By going through the derivation, argue that we can take
(1) the $e^{i\vec{q}\cdot\vec{r}}$ factor into account exactly, by replacing $\vec{p}_f$ by $\vec{p}_f - \hbar\vec{k}$ in Eq. 18.5.19

$$e^{-i\vec{p}_f\cdot\vec{r}/\hbar} \rightarrow e^{-i(\vec{p}_f - \hbar\vec{k})\cdot\vec{r}/\hbar} = e^{-i\vec{p}_f\cdot\vec{r}/\hbar}\, \underbrace{e^{+i\vec{k}\cdot\vec{r}}}$$

So the factor $e^{+i\vec{k}\cdot\vec{r}}$ is taken into account exactly.

(2) Verify the claim made above about the electron momentum distribution. Done in class...

$$H_\phi^0 = \frac{p^2}{2\mu} - \frac{Ze^2}{r} \quad \text{for Hydrogen-like atom}$$

$$H = \frac{(\vec{p} + \frac{q}{c}\vec{A})^2}{2\mu} - \frac{Ze^2}{r}$$ for Hydrogen-like atom in presence of a field (can ignore the $q\phi$ term)

$$H = \frac{p^2}{2\mu} + \frac{q\vec{A}\cdot\vec{p}}{\mu c} + \underbrace{\frac{q^2}{2\mu c^2}A^2} - \frac{Ze^2}{r}$$

$\vec{A}$ is assumed to be small, so this term is neglected to first order approximation.

$$H = H_\phi^0 + \underbrace{\frac{q}{\mu c}}_{V \text{ or } H'}\vec{A}\cdot\vec{p}$$

Let $\vec{A} = \vec{A}_0\cos(\vec{k}\cdot\vec{x} - \omega t)$ to represent light wave ($\vec{A}_0$: constant in time + space)

$$V = \frac{q}{\mu c}\vec{A}\cdot\vec{p} = \frac{q}{\mu c}\vec{A}_0\cos(\vec{k}\cdot\vec{x} - \omega t)\cdot\vec{p}$$

$$V = \frac{e}{2\mu c}\vec{A}_0\cdot\vec{p}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad \text{for absorption}$$

$$w_{fi}^{absorb} = \frac{2\pi}{\hbar}|{}_0\langle f|V_0|i\rangle_0|^2\, \delta(E_f^0 - E_i^0 - \hbar\omega)$$

where $V_0 = \frac{q}{2\mu c}e^{i\vec{k}\cdot\vec{x}}(\vec{A}_0\cdot\vec{p})$ time independent part of perturbation

$\langle\Psi_{before}|$ = ground state H atom $= {}_0\langle i| \Rightarrow \langle\vec{x}|i\rangle_0 = \sqrt{\frac{Z^3}{\pi a_0^3}}\, e^{-Zr/a_0}$

$\langle\Psi_{after}|$ = plane wave to represent unbound state of H atom $= {}_0\langle f| \Rightarrow \langle\vec{x}|f\rangle_0 = \sqrt{\frac{1}{2\pi\hbar}}\, e^{i\vec{p}\cdot\vec{x}/\hbar}$

$$H^0|\Psi_{before}\rangle = -Z^2 R_y = -Z^2(13.6\text{eV}) = E_i^0|\Psi_{before}\rangle$$

$$H^0|\Psi_{after}\rangle = \frac{p_f^2}{2\mu} = E_f^0|\Psi_{after}\rangle$$

$\vec{p}_f$, $\vec{A}_0$ $\vec{k}$ are all constants of integration $\quad Z=1$ for hydrogen

let $\vec{p} \to \frac{\hbar}{i}\vec{\nabla}$

$${}_0\langle f|V_0|i\rangle_0 = \frac{q}{2\mu c}\frac{1}{\sqrt{2\pi\hbar}^3}\frac{1}{\sqrt{\pi a_0^3}}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{-(i/\hbar)\vec{p}_f\cdot\vec{x}}\, e^{i\vec{k}\cdot\vec{x}}\, \vec{A}_0\cdot\frac{\hbar}{i}\vec{\nabla}\left(e^{-r/a_0} r^{-2}\right) dr \sin\theta d\theta d\phi$$

integrate by parts

$${}_0\langle f|V_0|i\rangle_0 = \frac{-q}{2\mu c\pi\sqrt{2\hbar a_0^3}}\frac{\hbar}{i}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\vec{\nabla}\left(e^{-(i/\hbar)(\vec{p}_f-\hbar\vec{k})\cdot\vec{x}}\right)\cdot\vec{A}_0\, e^{-r/a_0} r^2 dr \sin\theta d\theta d\phi$$

$\vec{\nabla}e^{i\vec{p}'\cdot\vec{x}} = i\vec{p}'\, e^{i\vec{p}'\cdot\vec{x}}$

0 because $\vec{k}\cdot\vec{A}_0 = 0$ Coulomb Gauge result

$${}_0\langle f|V_0|i\rangle_0 = \frac{+q\, 2\pi}{2\mu c\pi\sqrt{2\hbar a_0^3}}\int_0^{\pi}\int_0^{\infty}(\vec{p}_f-\hbar\vec{k})\cdot\vec{A}_0\, e^{-(i/\hbar)(\vec{p}_f-\hbar\vec{k})\cdot\vec{x}}\, e^{-r/a_0} r^2 dr \sin\theta d\theta$$

$(\vec{p}_f-\hbar\vec{k})\cdot\vec{x} = |\vec{p}_f-\hbar\vec{k}|r\cos\theta$

$\vec{p}_f\cdot\vec{A}_0$ is a constant of the integration

$${}_0\langle f|V_0|i\rangle_0 = \frac{+q\,\vec{p}_f\cdot\vec{A}_0}{\mu c\sqrt{2\hbar a_0^3}}\int_0^{\pi}\int_0^{\infty} e^{-i|\vec{p}_f-\hbar\vec{k}|r\cos\theta}\, e^{-r/a_0} r^2 dr \sin\theta d\theta$$

$${}_0\langle f|V_0|i\rangle_0 = \frac{-q}{\mu c}\frac{\vec{p}_f\cdot\vec{A}_0}{\sqrt{2\hbar a_0^3}}\int_0^{\infty}\int_0^{\pi} e^{-i|\vec{p}_f-\hbar\vec{k}|r\cos\theta}\, e^{-r/a_0} r^2 dr\, d\cos\theta$$

let $x = \cos\theta$

$${}_0\langle f|V_0|i\rangle_0 = \frac{+q}{\mu c}\frac{\vec{p}_f\cdot\vec{A}}{\sqrt{2\hbar a_0^3}}\int_0^{\infty}\int_{-1}^{+1} e^{i|\vec{p}_f-\hbar\vec{k}|rx} dx\, e^{-r/a_0} r^2 dr$$

let $v = -i|\vec{p}_f-\hbar\vec{k}|rx$

$${}_0\langle f|V_0|i\rangle_0 = \frac{+q}{\mu c}\frac{\vec{p}_f\cdot\vec{A}_0}{\sqrt{2\hbar a_0^3}}\int_0^{\infty} e^{-r/a_0} r^2 dr \int_{-i|\vec{p}_f-\hbar\vec{k}|r}^{+i|\vec{p}_f-\hbar\vec{k}|r}\frac{e^{vx}\, dv}{+i|\vec{p}_f-\hbar\vec{k}|r}$$

$${}_0\langle f|V_0|i\rangle_0 = \frac{+q}{\mu c}\frac{\vec{p}_f\cdot\vec{A}_0}{\sqrt{2\hbar a_0^3}}\int_{-\infty}^{+\infty} e^{-r/a_0} r^2 dr\left[\frac{e^{i|\vec{p}_f-\hbar\vec{k}|r} - e^{-i|\vec{p}_f-\hbar\vec{k}|r}}{i|\vec{p}_f-\hbar\vec{k}|r}\right]$$

$${}_0\langle f|V_0|i\rangle_0 = \frac{+q}{\mu c}\frac{\vec{p}_f\cdot\vec{A}_0}{\sqrt{2\hbar a_0^3}}\int_{-\infty}^{+\infty}\left(r e^{(i|\vec{p}_f-\hbar\vec{k}|-1/a_0)r} - r e^{(-i|\vec{p}_f-\hbar\vec{k}|-1/a_0)r} dr\right)\frac{1}{i|\vec{p}_f-\hbar\vec{k}|}$$

NOTE BOOK

let $u = \left[i|\vec{p}_f - \hbar \vec{k}| - 1/a_0\right] r \qquad v = \left[-i|\vec{p}_f - \hbar\vec{k}| - 1/a_0\right] r$

$${}_0\langle f | V_0 | i \rangle_0 = \frac{+q}{\mu c} \frac{\vec{p}_f \cdot \vec{A}_0}{\sqrt{2\hbar a_0^3}} \frac{1}{i|\vec{p}_f - \hbar\vec{k}|} \int_{-\infty}^{+\infty} \frac{u e^{u}\, du}{(i|\vec{p}_f - \hbar\vec{k}| - 1/a_0)^2} + \frac{v e^{v}\, dv}{[-i|\vec{p}_f - \hbar\vec{k}| - 1/a_0]^2}$$

$$\int_{-\infty}^{+\infty} u e^{u}\, du = \Gamma(2) = 1$$

$${}_0\langle f | V_0 | i \rangle_0 = \frac{+q}{\mu c} \frac{\vec{p}_f \cdot \vec{A}_0}{\sqrt{2\hbar a_0^3}} \frac{1}{i|\vec{p}_f - \hbar\vec{k}|} \left[\frac{1}{(i|\vec{p}_f - \hbar\vec{k}| - 1/a_0)^2} + \frac{1}{(-i|\vec{p}_f - \hbar\vec{k}| - 1/a_0)^2} \right]$$

$$\frac{2\pi}{\hbar} \left| {}_0\langle f | V_0 | i \rangle_0 \right|^2 = (\ )^2$$

Not complete!

-1

18.5.2 1) $\sigma = \frac{128 a_0^3 \pi q^2 p_f^3}{3m\hbar^3 \omega c[1+p_f^2 a_0^2/\hbar^2]^4}$

$\langle T \rangle = \frac{p_f^2}{2m} = 10 R_y \qquad \hbar\omega = \frac{p_f^2}{2m} - E_i^0 = 10R_y - (-1R_y) = 11 R_y$

$\sigma = \frac{128 a_0^3 \pi e^2 p_f^3}{3m\hbar^3 \omega c[1+(p_f a_0/\hbar)^2]^4} = \frac{128\pi}{3} \frac{p_f^3 a_0 e^2}{m\hbar^2} \frac{a_0^2}{(\hbar\omega c)} \frac{1}{[1+(p_f a_0/\hbar)^2]^4}$

$\frac{p_f^2}{2m} = \frac{p_f^3 p_f}{4m^2} = \frac{p_f^3}{m}\left(\frac{p_f}{4m}\right) = \frac{p_f^3}{m}\left(\frac{p_f}{4}\right)\left(\frac{a_0 e^2}{\hbar^2}\right) = \frac{p_f^3 a_0 e^2}{m\hbar^2}\left(\frac{p_f}{4}\right)$

$\sigma = \frac{128\pi}{3} \frac{4}{p_f} \left(\frac{p_f^2}{2m}\right)^2 \frac{a_0^2}{c} \left(\frac{1}{\hbar\omega}\right) \frac{1}{[1+p_f^2 a_0^2/\hbar^2]^4}$

$\sigma = \frac{128\pi 4}{3} \frac{(10R_y)^2}{(11R_y)} \frac{a_0^2}{p_f c} \frac{1}{[1+p_f^2 a_0^2/\hbar^2]^4}$

$\frac{p_f^2 a_0^2}{\hbar^2} = \frac{2mE_f a_0^2}{\hbar^2} = \frac{20 m a^2 R_y}{\hbar^2} = \frac{20 m a^2 m e^2}{2\hbar^4} = \frac{10 m^2 a^2 e^4}{\hbar^4} = 10$

$\sigma = \frac{128\pi 4 (10^2)}{3(11)} \frac{1}{[1+10]^4} \frac{a_0^2}{p_f c}$

$\sigma = \frac{128\pi 4(10^2)}{3(11)} \frac{1}{11^4} \left(\frac{\hbar}{p_f a_0}\right)\left(\frac{a_0^3}{\hbar c}\right)$

$\sigma = \frac{128\pi 4(10^2)}{3(11)^5 \sqrt{10}} \frac{a_0^3}{\hbar c} \rightarrow 10^{-3}\pi a_0^2$

NOTE BOOK

$$\left.\frac{d\sigma}{d\Omega}\right|_{H,\, Z=1} = \frac{32\, e^2 \hbar^5 \cos^2\theta}{m c \omega p_f^5 a_0^5}$$

$$\frac{p^2}{2m} \sim \frac{Ze^2}{r} \Rightarrow \frac{\hbar^2}{ma^2} \sim \frac{Ze^2}{a} \Rightarrow a \sim \frac{\hbar^2}{me^2}\frac{1}{Z} \qquad p \sim \frac{Zme^2}{\hbar}$$

$$E \sim \frac{Ze^2}{a} \sim \frac{Z^2 e^4 m}{\hbar^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{32\, e^2 \hbar^5 \cos^2\theta}{m c \omega p_f^5 \left(\frac{a_0}{Z}\right)^5} \sim Z^5$$