3/30/09

Update – after lecture covered material   Meg Noah

17.2.7 For the oscillator, consider $H' = -qfX$. Find Ω that satisfies 17.2.38. Feed to 17.2.39 for E_n^2 and compare with the earlier calculation.

$H^0 = P^2/2m + \frac{1}{2}m\omega^2 X^2$

$H' = [\Omega, H^0] = \Omega H^0 - H^0\Omega = \Omega\, P^2/2m + \Omega(\frac{1}{2}m\omega^2X^2) - P^2/2m\,\Omega - (\frac{1}{2}m\omega^2X^2)\Omega$

$H' = \Omega(P^2/2m) - (P^2/2m)\Omega + \Omega(\frac{1}{2}m\omega^2X^2) - \cancel{\Omega(\frac{1}{2}m\omega^2X^2)}$

$H' = \Omega\left(\frac{-\hbar^2}{2m}\right)\frac{\partial^2}{\partial x^2} - \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Omega\right) = -qfX$

$H' = -\frac{\hbar^2}{2m}\Omega\frac{\partial^2}{\partial x^2} + \frac{\hbar^2}{2m}\frac{\partial^2\Omega}{\partial x^2} + \frac{\hbar^2}{2m}\Omega\frac{\partial^2}{\partial x^2} = -qfX$

$\frac{\partial^2\Omega}{\partial x^2} = +\frac{qf\,2m}{\hbar^2}X = \frac{\partial^2\Omega}{\partial X^2}$

$\cancel{\frac{\partial^2\Omega}{\partial X^2} + \left(\frac{qf2m}{\hbar^2}\right)X = 0}$

$\cancel{X = e^{-iqf2mX/\hbar^2}}$

$\boxed{\Omega = +\frac{qf2m\,X^3}{\hbar^2 3\cdot 2}} \Rightarrow \frac{\partial\Omega}{\partial X} = +\frac{qf2mX^2}{\hbar^2 2} \quad \frac{\partial^2\Omega}{\partial X^2} = +\frac{qf2m\,X}{\hbar^2}$

$E_n^2 = \sum_m{}' \frac{\langle n^0|H'|m^0\rangle\langle m^0|\Omega H^0 - H^0\Omega|n^0\rangle}{E_n^0 - E_m^0} = \langle n^0|H'\Omega|n^0\rangle - \langle n^0|H'|n^0\rangle\langle n^0|\Omega|n^0\rangle$

$\langle n^0|H'|n^0\rangle = \langle n^0|-qfX|n^0\rangle = 0$ because $\langle n|X|n\rangle = 0 \quad X \propto (a^+ + a^-)$

$\langle n^0|\Omega|n^0\rangle = \langle n^0|\left(\frac{+qf\,m}{3\hbar^2}\right)(X^3)|n^0\rangle = +\frac{qfm}{3\hbar^2}\langle n^0|X^3|n^0\rangle = 0$

$X^3 \propto (a^3 + a^2a^+ + aa^+a + aa^+a^+ + a^+aa + a^+aa^+ + a^+a^+a + a^+a^+a^+)$ P. 209 $\rightarrow \langle n^0|X^3|n^0\rangle = 0$

$\langle n^0|H'\Omega|n^0\rangle = (-qf)\left(\frac{+qfm}{3\hbar^2}\right)\langle n^0|X^4|n^0\rangle = \frac{q^2f^2m}{3\hbar^2}\langle n^0|X^4|n^0\rangle = \cancel{\frac{q^2f^2m}{3\hbar^2}}$

$X^4 = \left(\frac{\hbar}{2m\omega}\right)^2(a^4 + a^3a^+ + a^2a^+a + aaa^+a^+ + aa^+aa + aa^+aa^+ + aa^+a^+a + aa^+a^+a^+$
$+ a^+a^3 + a^+a^2a^+ + a^+aa^+a + a^+aa^+a^+ + a^+a^+aa + a^+a^+aa^+ + a^+a^+a^+a^+ + a^{+4})$

terms contributing

$X^4 = \left(\frac{\hbar}{2m\omega}\right)^2(aaa^+a^+ + aa^+aa^+ + aa^+a^+a + a^+aaa^+ + a^+aa^+a + a^+a^+aa)$

$E_n^2 = \langle n^0|H'\Omega|n^0\rangle = -\frac{q^2f^2m}{3\hbar^2}\left(\frac{\hbar^2}{4m^2\omega^2}\right)\langle n^0|aaa^+a^+ + aa^+aa^+ + aa^+a^+a + a^+aaa^+ + a^+aa^+a + a^+a^+aa|n^0\rangle$

$$E_n^2 = \frac{-q^2 f^2}{12 m\omega^2} \langle n^0 | aaa^+a^+ + aa^+aa^+ + aa^+a^+a + a^+a^+aa + a^+aaa^+ + a^+aa^+a | n^0 \rangle$$

$$a^+a|n\rangle = n|n\rangle \qquad aa^+|n\rangle = a(n+1)^{1/2}|n+1\rangle = (n+1)|n\rangle$$

$$aa|n\rangle = (n-1)^{1/2} n^{1/2} |n-2\rangle \qquad a^+a^+ = (n+1)^{1/2} \cancel{(n)^{1/2}} \underset{(n+2)^{1/2}}{} |n+2\rangle$$

$$E_n^2 = \frac{-q^2 f^2}{12 m\omega^2} \Big[\langle n^0 | aa (n+1)^{1/2} (n+2)^{1/2} | n^0 + 2 \rangle + \langle n|(n+1) aa^+ |n\rangle + \langle n | aa^+ n | n \rangle$$

$$+ \langle n | a^+a^+ n^{1/2} (n-1)^{1/2} | n-2 \rangle + \langle n | a^+a (n+1) | n \rangle + \langle n | a^+a\, n | n \rangle \Big]$$

$$E_n^2 = \frac{-q^2 f^2}{12 m\omega^2} \Big[\overset{(n+1)(n+2)}{\cancel{(n+2)(n+2)}} + (n+1)^2 + n(n+1) + n(n-1) + n(n+1) + n^2 \Big]$$

$$E_n^2 = \frac{-q^2 f^2}{12 m\omega^2} \Big[n^2 + n + 2n + 2 + n^2 + 2n + 1 + n^2 + n + n^2 - n + n^2 + n + n^2 \Big]$$

$$E_n^2 = \frac{-q^2 f^2}{12 m\omega^2} \Big[6n^2 + \underset{6n}{\cancel{8n}} + 3 \Big]$$

~~looks the same~~

$$E_n^2 = \frac{-q^2 f^2}{4 m\omega^2} \Big[2n^2 + 2n + 1 \Big]$$

previously got

$$E_n^2 = -\frac{q^2 f^2}{2 m\omega^2}$$

3/30/09

Update after last lecture covered material   Meg Noah

17.2.6 Shankar Verify $|\langle 210|z|100\rangle|^2 = \frac{2^{15}}{3^{10}} a_0 \approx .55 a_0^2$

$$\cancel{E_0^0 - E_{nlm}^0 = -Ry\left(1-\frac{1}{n^2}\right) = Ry\left(\frac{1-n^2}{n^2}\right)}$$

$$\cancel{E_0^0 - E_{210}^0 = -Ry\left(1-\frac{1}{4}\right) = -\frac{3}{4}Ry}$$

$$z = r\cos\theta \qquad d\tau = r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$\psi_{100} = \left(\frac{1}{\pi a_0^3}\right)^{1/2} e^{-r/a_0} \qquad \psi_{210} = \left(\frac{1}{32\pi a_0^3}\right)^{1/2} \frac{r}{a_0} e^{-r/2a_0} \cos\theta$$

$$\langle 210|z|100\rangle = \int \psi_{210}\, r\cos\theta\, \psi_{210}\, d\tau$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(\frac{1}{\pi a_0^3}\right)^{1/2} e^{-r/a_0}\, r\cos\theta \left(\frac{1}{32\pi a_0^3}\right)^{1/2} \frac{r}{a_0} e^{-r/2a_0} \cos\theta\, r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$= \frac{1}{\sqrt{32}\,\pi a_0^3} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{r^4}{a_0} e^{-r/a_0 - r/2a_0} dr \cos^2\theta \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{\sqrt{32}\,\pi a_0^3} \frac{(2\pi)}{a_0} \int_0^{\infty} r^4 e^{-3r/a_0} dr \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta$$

$\theta\{0,\pi\}$
$\mu = \cos\theta$
$d\mu = -\sin\theta\, d\theta$
$\mu = \{1,-1\}$

$$= \frac{1}{\sqrt{32}\,\pi a_0^3} \frac{(2\pi)}{a_0} \int_0^{\infty} r^4 e^{-3r/a_0} dr \int_{-1}^{+1} \mu^2 d\mu$$

$$\cancel{= \frac{1}{\sqrt{32}\,\pi a_0^3}(2\pi) \frac{\Gamma(4+1)}{(-3/a_0)^5} \left(\frac{\mu^3}{3}\right)\Big|_{-1}^{+1} = \frac{1}{\sqrt{32}\, a_0^3}(2)\, a_0^5 \frac{(120)}{3^5}\frac{2}{3} = \frac{4 a_0^2}{\sqrt{32}}\frac{40}{3^5}}$$

$\Gamma(5) = 4\Gamma(4) = 4\cdot 3\Gamma(3) = 4\cdot 3\cdot 2\Gamma(2)$

$\cancel{\Gamma(5) = \Gamma(4) = \Gamma(3) = 60\Gamma(2) = 120\Gamma(1) = 120}$ $\Gamma(5) = 4\cdot 3\cdot 2$

$= 4\cdot 3\cdot 2\cdot 1\Gamma(1)$

$$\cancel{= \frac{2\cdot a_0^4\, 4\cdot 3\cdot 2\cdot 2}{\sqrt{32}\, a_0^3\, 3^5} = \frac{2\, a_0\, 4\cdot 3\cdot 2\cdot 2}{\sqrt{16}\sqrt{2}\, 3^5} = \frac{2^3}{\sqrt{2}}\frac{a_0}{3^4}}$$

$$= \frac{2}{4\sqrt{2}\, a_0^4} \underbrace{\int_0^{\infty} r^4 e^{-3r/a_0} dr}_{\frac{\Gamma(5)}{(-3/a_0)^5}} \underbrace{\int_{-1}^{+1} \mu^2 d\mu}_{2/3} = \frac{-4\cdot 3\cdot 2\, a_0}{\sqrt{2}\, 3^5\, 3}$$

$-1/2$

$$\langle 210|Z|100\rangle = -\frac{2^3 a_0}{\sqrt{2}\, 3^5}$$

I can't get the factor of 2^8 here - maybe the normalization isn't right?

Should get $\langle 210|z|100\rangle = -\frac{2^8 a_0}{\sqrt{2}\, 3^5}$