1.5 (page 16) Show that the z component of angular momentum for a point particle $L_z = xp_y - yp_x$ when expressed in spherical coordinates becomes $L_z = p_\phi = mr^2\dot{\phi}\sin^2\theta$.

$$x = r\cos\phi\sin\theta$$
$$y = r\sin\phi\sin\theta$$
$$\dot{x} = \dot{r}(\cos\phi\sin\theta) + r(-\sin\phi)\dot{\phi}\sin\theta + r\cos\phi\cos\theta\dot{\theta}$$
$$\dot{y} = \dot{r}(\sin\phi\sin\theta) + r(\cos\phi)\dot{\phi}\sin\theta + r\sin\phi\cos\theta\dot{\theta}$$
$$\begin{aligned} xp_y - yp_x &= xm\dot{y} - ym\dot{x} \\ &= mr\cos\phi\sin\theta\left(\dot{r}(\sin\phi\sin\theta) + r(\cos\phi)\dot{\phi}\sin\theta + r\sin\phi\cos\theta\dot{\theta}\right) - \\ &\qquad mr\sin\phi\sin\theta\left(\dot{r}(\cos\phi\sin\theta) + r(-\sin\phi)\dot{\phi}\sin\theta + r\cos\phi\cos\theta\dot{\theta}\right) \\ &= mr^2\left((\cos^2\phi)\dot{\phi}\sin^2\theta + (\sin^2\phi)\dot{\phi}\sin^2\theta\right) \\ &= mr^2\dot{\phi}\sin^2\theta \end{aligned}$$

1.7 (p 18) Show that the energy of a free particle may be written $H = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2}$ where $\vec{L} = \vec{r}\times\vec{p}$. Hint: Use the vector relation $\vec{L}^2 = (\vec{r}\times\vec{p})^2 = r^2p^2 - (\vec{r}\cdot\vec{p})^2$ together with the definition $p_r = (\vec{r}\cdot\vec{p})/r$.

Start with the Hamiltonian for a free particle in spherical coordinates:

$$p_r = \hat{r}\cdot\vec{p} = \frac{1}{r}\vec{r}\cdot\vec{p} \Rightarrow \vec{r}\cdot\vec{p} = rp_r$$
$$H = \frac{m}{2}v^2 = \frac{p^2}{2m}$$
$$L^2 = (\vec{r}\times\vec{p})^2 = r^2p^2 - (\vec{r}\cdot\vec{p})^2 = r^2p^2 - r^2p_r^2 \Rightarrow \frac{L^2}{r^2} = p^2 - p_r^2$$
$$\Rightarrow p^2 = \frac{L^2}{r^2} + p_r^2$$
$$H = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2}$$

1.8 (p 18) Show that angular momentum of a free particle obeys the relation $L^2 = L_x^2 + L_y^2 + L_z^2 = p_\theta^2 = \frac{p_\phi^2}{\sin^2\theta}$.

$H = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2}$ (from last problem)

$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}$ (from book)

subtract $\rightarrow \frac{L^2}{r^2} = \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta} \rightarrow L^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}$

1.21 (p. 26) Use the expression $e^{i\theta} = \cos\theta + i\sin\theta$ to derive the following relations:

① $e^{i(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1)}e^{i(\theta_2)} = (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2))$

② $e^{i(\theta_1-\theta_2)} = \cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2) = e^{i(\theta_1)}e^{-i(\theta_2)} = (\cos(\theta_1) + i\sin(\theta_1))(-\cos(\theta_2) - i\sin(\theta_2))$

③ $e^{2i(\theta)} = \cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2$

a) $\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$ (from real part ①)

b) $\sin(\theta_1+\theta_2) = \cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2$ (from imaginary part ①)

c) $\sin(\theta_1-\theta_2) + \sin(\theta_1+\theta_2) = 2\sin\theta_1\cos\theta_2$ (from Im①+Im②)

d) $\cos(\theta_1-\theta_2) + \cos(\theta_1+\theta_2) = 2\cos\theta_1\cos\theta_2$ (from Re①+Re②)

e) $\cos(\theta_1-\theta_2) - \cos(\theta_1+\theta_2) = 2\sin\theta_1\sin\theta_2$ (from Re②-Re①)

f) $2\cos^2\theta = 1 + \cos 2\theta$ (from Re③)

g) $2\sin^2\theta = 1 - \cos 2\theta$ (from f with identity $\sin^2 + \cos^2 = 1$)

h) $e^{i\theta} - 1 = 2ie^{i\theta/2}\sin\theta/2$

$e^{i\theta} = e^{i\theta/2+i\theta/2} = \cos\theta + i\sin\theta = (\cos\theta/2 + i\sin\theta/2)(\cos\theta/2 + i\sin\theta/2)$

$e^{i\theta} = \underbrace{\cos^2\theta/2 + \sin^2\theta/2}_{1} + 2i\sin\theta/2\cos\theta/2 - 2\sin^2\theta/2$

$e^{i\theta} - 1 = 2i\sin\theta/2(\cos\theta/2 - \sin\theta/2) = 2i\sin\theta/2\, e^{i\theta/2}$

i) $\frac{1}{2}\left|e^{i\theta_1} + e^{i\theta_2}\right|^2 = \frac{1}{2}\left(e^{i\theta_1} + e^{i\theta_2}\right)\left(e^{i\theta_1} + e^{i\theta_2}\right)^* = 1 + \cos(\theta_1 - \theta_2)$

$= \frac{1}{2}\left(e^{i\theta_1} + e^{i\theta_2}\right)\left(e^{-i\theta_1} + e^{-i\theta_2}\right) = \frac{1}{2}\left(e^0 + e^{i(\theta_2-\theta_1)} + e^{i(\theta_1-\theta_2)} + e^0\right)$

$= \frac{1}{2}\left(2 + e^{i(\theta_2-\theta_1)} + e^{i(\theta_1-\theta_2)}\right)$

$= \frac{1}{2}\left(2 + 2\cos(\theta_1-\theta_2) - i\sin(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$

$= 1 + \cos(\theta_1 - \theta_2)$

j) $2\operatorname{Re} z = z + z^*$ $\quad z = e^{i\theta}$

$e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta = 2\operatorname{Re} z$

k) $2i\operatorname{Im} z = z - z^*$

$e^{i\theta} - e^{-i\theta} = \cos\theta + i\sin\theta - \cos\theta + i\sin\theta = 2i\sin\theta = 2\operatorname{Im} z$

l) $\left(e^{z}\right)^{*} = e^{z*}$

$\left(\exp(\cos\theta + i\sin\theta)\right)^* = \exp(\cos\theta - i\sin\theta) = \exp(e^{-i\theta}) = e^{z*}$

m) $\left|\exp z\right|^2 = \exp\left(2\operatorname{Re} z\right)$

$e^{z}\left(e^{z}\right)^* = e^{z}e^{z*} = e^{z+z*} = e^{(2\operatorname{Re} z)}$

n) $i^{i} = e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, \ldots i = e\left[i\pi\,\tfrac{1+4n}{2}\right]$

$i = \dfrac{e^{i\theta} - \cos\theta}{\sin\theta}$ where $\theta = \pi\,\tfrac{1+4n}{2}$ $\quad n = 0,1,2,\ldots$

$\sin\theta = \sin\tfrac{\pi}{2} = \sin\tfrac{5\pi}{2} = \sin\tfrac{9\pi}{2} = \cdots = 1$

$\cos\theta = \cos\tfrac{\pi}{2} = \cos\tfrac{5\pi}{2} = \cos\tfrac{9\pi}{2} = \cdots = 0$

$i = e\left[i\pi\,\tfrac{1+4n}{2}\right]$

$i^{i} = \left(e\left[i\pi\,\tfrac{1+4n}{2}\right]\right)^{i} = e^{-\pi\frac{1+4n}{2}}$

2.23 (page 50) Show that the de Broglie wavelength of an electron of kinetic energy E (eV) is $\lambda_e = \dfrac{1.23\cdot 10^{-8}}{\mathrm{E}^{1/2}}\,\mathrm{cm}$ and that of a proton is $\lambda_p = \dfrac{0.29\cdot 10^{-8}}{\mathrm{E}^{1/2}}\,\mathrm{cm}$.

$$\lambda_{deBroglie} = \frac{h}{p}$$

$$p = \sqrt{E2m}$$

$$\lambda_{deBroglie}^2 = \frac{h^2}{p^2} = \frac{h^2}{2Em}$$

$$\lambda = \sqrt{\frac{h^2}{2Em}} = \sqrt{\frac{h^2c^2}{2Em}} \text{ where rest mass in eV}$$

$$\lambda = \tfrac{1}{2}\left(4.1257\cdot 10^{-7}\,\mathrm{eVs}\right)\left(3\cdot 10^{8}\,\mathrm{ms^{-1}}\right)\frac{1}{\sqrt{m}}\frac{1}{\sqrt{E}}$$

$$\lambda_e = \tfrac{1}{2}\left(4.1257\cdot 10^{-7}\,\mathrm{eVs}\right)\left(3\cdot 10^{10}\,\mathrm{cms^{-1}}\right)\frac{1}{\sqrt{0.511\cdot 10^{6}\,\mathrm{eV}}}\frac{1}{\sqrt{E}} = \frac{1.227\cdot 10^{-7}\,\mathrm{cm}}{\sqrt{E}}$$

$$\lambda_p = \tfrac{1}{2}\left(4.1257\cdot 10^{-7}\,\mathrm{eVs}\right)\left(3\cdot 10^{10}\,\mathrm{cms^{-1}}\right)\frac{1}{\sqrt{938.27\cdot 10^{6}\,\mathrm{eV}}}\frac{1}{\sqrt{E}} = \frac{2.86\cdot 10^{-9}\,\mathrm{cm}}{\sqrt{E}}$$

Give a proof that the Hamiltonian and the linear momentum operators for a free particle have common eigenfunctions.

This proof shows that the contrary assumption leads to contradiction.

Let φ be the eigenfunction of the Hamiltonian operator and ϕ be the eigenfunction of the linear momentum operator.

	Hamiltonian	Linear Momentum
Operator	$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2$	$\hat{p} = -i\hbar\nabla$
Eigenvalus for Free Particle	$E = \frac{\hbar^2 k^2}{2m}$	$p = \hbar k$
Eigenvalue equation	$\hat{H}\varphi = E\varphi$	$\hat{p}\phi = p\phi$
Solution	$\nabla^2\varphi = -k^2\varphi$ $\frac{\nabla^2\varphi}{\varphi} = -k^2$	$-i\hbar\nabla\phi = \hbar k\phi$ $\nabla\phi = ik\phi$ $\nabla^2\phi = ik\nabla\phi$ $\nabla^2\phi = ik(ik\phi)$ $\frac{\nabla^2\phi}{\phi} = -k^2$
Eigenfunction	$\varphi = Ae^{ikx} + Be^{-ikx}$	$\phi = Ce^{ikx} + De^{-ikx}$

Combining the eigenfunctions both = $-k^2$:

$$\phi\nabla^2\varphi - \varphi\nabla^2\phi = 0$$

$$\phi\nabla\varphi - \varphi\nabla\phi = \text{constant}$$

Show the constant is equal to 0, by substituting the eigenfunctions and their derivatives:

$$\varphi\nabla\phi = \phi\nabla\varphi = ikACe^{ikx} + ikBDe^{-ikx} + ikAD - ikBC$$

$$\frac{\nabla\varphi}{\varphi} = \frac{\nabla\phi}{\phi}$$

Integrate to get $\ln(\phi) = \ln(\varphi) + \text{constant}$. This is $\phi = \text{constant}\times\varphi$. Therefore ϕ and φ represent the same state vector according to the statistical interpretation of the wavefunction.

3.6 Establish the following properties of δ(y):

(a) δ(y) = δ(-y)

δ(y)=0 when y≠0 and when y=0, y=-y therefore δ(y) = δ(-y)

Note: This is also seen from the symmetry in the delta function plot. This is a special case of part (d), which in turn is a special case of part (i). In part (d), setting a = -1, δ(ay) = δ(-y) = $|a|^{-1}$δ(y) = δ(y)

(b) δ'(y) = -δ'(-y)

δ'(y) = -δ'(−y)

−y δ'(y) = y δ'(−y)

−δ(y) = y δ'(−y)

δ(y) = − y δ'(−y)

$$\int_{-\infty}^{\infty} f(y)(-y)\delta'(-y)dy = -\int_{-\infty}^{\infty}\frac{d}{dy}(fy\delta)dy + \int_{-\infty}^{\infty}\delta\frac{d}{dy}(fy)dy = \int_{-\infty}^{\infty}\delta(y\frac{df}{dy}+f)dy = \int_{-\infty}^{\infty}\delta y f(y)dy$$

therefore:

δ(y) = − y δ'(−y) and δ'(y) = -δ'(−y)

(c) y δ(y) = 0

when y=0, y δ(y) = 0 when y≠0, δ(y) = 0, so y δ(y) = 0

(d) δ(ay)=$|a|^{-1}$δ(y)

let x = ay then dx=ady

$$\int \delta(x)dx = 1$$

$$\int \delta(ay)ady = 1$$

$$\int \delta(ay)dy = \frac{1}{a}$$

Since δ(y) = δ(-y) if a<0, $\int \delta(ay)dy = \int \delta(-ay)dy = \frac{1}{a} = \frac{1}{|a|}$

(e) $\delta(y^2-a^2)=|2a|^{-1}[\delta(y-a) + \delta(y+a)]$

For the LHS, let $x = y^2-a^2 = (y-a)(y+a)$. x=0 at two places a and –a
dx = 2ydy

Note: factor of 2 from the two roots.

$$\int_{-\infty}^{\infty}\delta(y^2-a^2)dy = 2\int\delta(x)\frac{dx}{2y} = \int\frac{1}{\sqrt{x+a^2}}\delta(x)dx = \frac{1}{|a|}$$

For the RHS:

$$\int_{-\infty}^{\infty}[\delta(y-a)+\delta(y+a)]dy = \int_{-\infty}^{\infty}\delta(y-a)dy + \int_{-\infty}^{\infty}\delta(y+a)dy = 2$$

$$\frac{1}{2}\int_{-\infty}^{\infty}[\delta(y-a)+\delta(y+a)]dy = 1 = |a|\int_{-\infty}^{\infty}\delta(y^2-a^2)dy$$

$$\int_{-\infty}^{\infty}\frac{1}{2|a|}[\delta(y-a)+\delta(y+a)]dy = \int_{-\infty}^{\infty}\delta(y^2-a^2)dy$$

$$\delta(y^2-a^2) = \frac{1}{2|a|}[\delta(y-a)+\delta(y+a)]$$

(f) $\int_{-\infty}^{\infty}\delta(a-y)\delta(y-b)\,dy = \delta(a-b)$

let f(y) = δ(a-y) then use $\int_{-\infty}^{\infty}f(y)\delta(y-b)dy = f(b)$

$$\int_{-\infty}^{\infty}\delta(a-y)\delta(y-b)dy = f(y=b) = \delta(a-b)$$

(g) $f(y)\delta(y-a) = f(a)\delta(y-a)$

$$\int_{-\infty}^{\infty}f(y)\delta(y-a)dy = \int_{-\infty}^{\infty}f(a)\delta(y-a)dy$$

$$f(a) = f(a)\int_{-\infty}^{\infty}\delta(y-a)dy$$

therefore $f(y)\delta(y-a) = f(a)\delta(y-a)$

(h) yδ'(y) = -δ(y)

$$\int_{-\infty}^{\infty} f(y) y\delta'(y) dy = \int_{-\infty}^{\infty} \frac{d}{dy}(fy\delta) dy - \int_{-\infty}^{\infty} \delta \frac{d}{dy}(yf) dy = -\int_{-\infty}^{\infty} \delta(y)\left(\delta \frac{df}{dy} + f\right) dy = -\int_{-\infty}^{\infty} \delta(y) f(y) dy$$

This establishes $y\delta'(y) = -\delta(y)$

3.7 Show that the following are valid representations of δ(y):

(a) $2\pi\delta(y) = \int_{-\infty}^{\infty} e^{iky} dy$

use the Fourier Transform definition of the dirac function $\int_{-\infty}^{\infty} 1 \cdot e^{-i2\pi f t} \, dt = \delta(f)$

substitute k = 2 π *f*

(b) $\pi\delta(y) = \lim_{\eta \to \infty} \left(\frac{\sin(\eta y)}{y} \right)$

$$\int_{-\infty}^{\infty} \pi\delta(y) dy = \int_{-\infty}^{\infty} \left(\frac{\sin(\eta y)}{y} \right) dy$$

$$\int_{0}^{\infty} \left(\frac{\sin(my)}{y} \right) dy = \frac{\pi}{2} \text{ if } m > 0; 0 \text{ if } m = 0; -\frac{\pi}{2} \text{ if } m < 0 \text{ so } \int_{-\infty}^{\infty} \left(\frac{\sin(my)}{y} \right) dy = \pi \text{ for } m \to \infty$$

$$\pi \int_{-\infty}^{\infty} \delta(y) dy = \pi$$

3.11 Calculate the uncertainty Δp for a particle in the state ψ given by (3.37). Do you find your answer to be consistent with the uncertainty principle? (In this problem one must calculate $\langle \hat{p}^2 \rangle$. The operator: $\hat{p}^2 = -i\hbar \partial^2 / \partial x^2$.)

$$\langle \Delta p \rangle^2 = \langle \hat{p}^2 \rangle - \langle p \rangle^2$$
$$\langle \Delta p \rangle = \sqrt{\langle \hat{p}^2 \rangle - \langle p \rangle^2}$$

Using equations 3.37, 3.38, and 3.39

$$\psi(x,t) = A\exp\left[\frac{-(x-x_0)^2}{4a^2}\right]\exp\left(\frac{ip_0 x}{\hbar}\right)\exp(-i\omega_0 t)$$

$$A^2 = \frac{1}{a\sqrt{2\pi}}$$

introduce dummy variables η and η_0

$$\eta = \frac{(x-x_0)}{a}$$
$$x = a(\eta + \eta_0)$$
$$\eta_0 = \frac{x_0}{a}$$

Done in book, equation (3.44)

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi * \hat{p}\psi dx = \int_{-\infty}^{\infty} \psi * \left(-i\hbar \frac{\partial}{\partial x}\right)\psi dx = A^2 a \int_{-\infty}^{\infty} \left(p_0 + \frac{i\hbar}{2a}\eta \right) e^{-\eta^2/2} d\eta$$

$$= p_0 A^2 a \int_{-\infty}^{\infty} e^{-\eta^2/2} d\eta = p_0 A^2 a\sqrt{2\pi} = p_0$$

Now find <p^2>

$$\left\langle p^2\right\rangle = \int_{-\infty}^{\infty} \psi * \hat{p}\hat{p}\psi \, dx = \int_{-\infty}^{\infty} \psi * \left(-i\hbar\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi \, dx$$

$$= -\hbar^2 \int_{-\infty}^{\infty} \psi * \left(\frac{\partial^2}{\partial x^2}\right)\psi \, dx$$

$$= -\hbar^2 \int_{-\infty}^{\infty} \psi * \left(\frac{\partial^2}{\partial x^2}\right) A\exp\left[\frac{-(x-x_0)^2}{4a^2}\right]\exp\left(\frac{ip_0x}{\hbar}\right)\exp(-i\omega_0 t)\, dx$$

$$= -\hbar^2 \exp(-i\omega_0 t)\int_{-\infty}^{\infty} \psi * \left(\frac{\partial^2}{\partial x^2}\right) A\exp\left[\frac{-(x-x_0)^2}{4a^2}\right]\exp\left(\frac{ip_0x}{\hbar}\right) dx$$

$$= -\hbar^2 A\exp(-i\omega_0 t)\int_{-\infty}^{\infty} \psi * \left(\frac{\partial}{\partial x}\right)\left\{-\frac{(x-x_0)}{2a^2}+\frac{ip_0}{\hbar}\right\}\exp\left[\frac{-(x-x_0)^2}{4a^2}\right]\exp\left(\frac{ip_0x}{\hbar}\right) dx$$

$$= -\hbar^2 \int_{-\infty}^{\infty} \psi * \left(\frac{\partial}{\partial x}\right)\left\{-\frac{(x-x_0)}{2a^2}+\frac{ip_0}{\hbar}\right\}\psi \, dx$$

$$= -\hbar^2 \int_{-\infty}^{\infty} \psi * \left\{-\frac{1}{2a^2}\right\}\psi + \left\{-\frac{(x-x_0)}{2a^2}+\frac{ip_0}{\hbar}\right\}\left(\frac{\partial \psi}{\partial x}\right) dx$$

$$= -\hbar^2 \int_{-\infty}^{\infty} \psi * \left\{-\frac{1}{2a^2}\right\}\psi + \left\{-\frac{(x-x_0)}{2a^2}+\frac{ip_0}{\hbar}\right\}\left\{-\frac{(x-x_0)}{2a^2}+\frac{ip_0}{\hbar}\right\}\psi \, dx$$

$$= -\hbar^2 \int_{-\infty}^{\infty} \left[-\frac{1}{2a^2}+\left\{-\frac{(x-x_0)}{2a^2}+\frac{ip_0}{\hbar}\right\}^2\right]\psi * \psi \, dx$$

$$= -\hbar^2 A^2 \int_{-\infty}^{\infty} \left[-\frac{1}{2a^2}+\left\{-\frac{(x-x_0)}{2a^2}+\frac{ip_0}{\hbar}\right\}^2\right]\exp\left[\frac{-(x-x_0)^2}{2a^2}\right] dx$$

$$= -\hbar^2 A^2 \int_{-\infty}^{\infty} \left[-\frac{1}{2a^2}+\left(\frac{(x-x_0)}{2a^2}\right)^2 - 2\frac{ip_0}{\hbar}\frac{(x-x_0)}{2a^2}-\left(\frac{p_0}{\hbar}\right)^2\right]\exp\left[\frac{-(x-x_0)^2}{2a^2}\right] dx$$

$$= -\hbar^2 A^2 \left[-\frac{1}{2a^2}\sqrt{2\pi}a + \left(\frac{1}{2a^2}\right)^2\sqrt{2\pi}a^3 - 0 - \left(\frac{p_0}{\hbar}\right)^2\sqrt{2\pi}a\right]$$

$$= -\hbar^2 A^2 \left[-\frac{1}{2a^2}\sqrt{2\pi}a + \left(\frac{1}{2a^2}\right)^2\sqrt{2\pi}a^3 - 0 - \left(\frac{p_0}{\hbar}\right)^2\sqrt{2\pi}a\right]$$

$$= \hbar^2\left[\frac{1}{2a^2}-\frac{1}{4a^2}+\left(\frac{p_0}{\hbar}\right)^2\right]$$

$$= \frac{\hbar^2}{4a^2}+p_0^2$$

$$\left\langle p\right\rangle = p_0$$

$$\left\langle p\right\rangle^2 = p_0^2$$

$$\left\langle p^2\right\rangle = \frac{\hbar^2}{4a^2}+p_0^2$$

$$\Delta p = \sqrt{\left\langle p^2\right\rangle - \left\langle p\right\rangle^2} = \sqrt{\frac{\hbar^2}{4a^2}+p_0^2-p_0^2} = \frac{\hbar}{2a}$$

$$\Delta x = \sqrt{\left\langle x^2\right\rangle - \left\langle x\right\rangle^2} = \sqrt{a^2+x_0^2-x_0^2} = a$$

$$\Delta x \Delta p = a\frac{\hbar}{2a} = \frac{\hbar}{2}$$

This is consistent with Heisenberg's uncertainty principle.

4.1 What are the energy eigenfunctions and eigenvalues for the one-dimensional box problem described above if the ends of the box are at –a/2 and +a/2? [Check you answer with (6.100).]

$$\hat{H}_1 = -\frac{\hbar^2}{2m}\nabla^2 + \infty = \infty \qquad \text{for } \left(\frac{\mathrm{a}}{2} \le \mathrm{x} \text{ or } \mathrm{x} \le -\frac{a}{2}\right)$$

$$\hat{H}2 = -\frac{\hbar^2}{2m}\nabla^2 \qquad \text{for } \left(-\frac{\mathrm{a}}{2} \le \mathrm{x} \le +\frac{a}{2}\right)$$

boundary conditions from regions outside the well are:

$$\varphi\left(\frac{\mathrm{a}}{2}\right) = \varphi\left(-\frac{\mathrm{a}}{2}\right) = 0$$

$$\hat{H}_2\varphi = E\varphi$$

$$\varphi_n = A\sin k_n x + B\cos k_n x$$

$$\varphi_n\left(\frac{\mathrm{a}}{2}\right) = A\sin k_n \frac{\mathrm{a}}{2} + B\cos k_n \frac{\mathrm{a}}{2} = 0$$

$$\varphi_n\left(-\frac{\mathrm{a}}{2}\right) = A\sin k_n \frac{\text{-a}}{2} + B\cos k_n \frac{\text{-a}}{2} = -A\sin k_n \frac{\mathrm{a}}{2} + B\cos k_n \frac{\mathrm{a}}{2} = 0$$

repeating

$$\varphi_n\left(\frac{\mathrm{a}}{2}\right) = A\sin k_n \frac{\mathrm{a}}{2} + B\cos k_n \frac{\mathrm{a}}{2} = 0$$

$$\varphi_n\left(-\frac{\mathrm{a}}{2}\right) = A\sin k_n \frac{\text{-a}}{2} + B\cos k_n \frac{\text{-a}}{2} = -A\sin k_n \frac{\mathrm{a}}{2} + B\cos k_n \frac{\mathrm{a}}{2} = 0$$

where

$$k^2 = \frac{2mE}{\hbar^2}$$

$$k_n x = n\pi$$

$$\frac{k_n x}{2} = \frac{n\pi}{2}$$

$$\sin k_n \frac{\mathrm{x}}{2} = \sin k_n \frac{\mathrm{a}}{2} = \sin k_n \frac{n\pi}{2} = 0 \quad \text{when n} = 2,4,6,...$$

$$\cos k_n \frac{\mathrm{x}}{2} = \cos k_n \frac{\mathrm{a}}{2} = \cos k_n \frac{n\pi}{2} = 0 \quad \text{when n} = 1,3,5,...$$

$$k_n = \frac{n\pi}{a} = \sqrt{\frac{2mE}{\hbar^2}}$$

$$E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2\pi^2}{2ma^2}n^2 \quad \text{when n} = 1,3,5,...$$

Find the amplitude of the energy eigenfunction by normalization

$$<\varphi|\varphi>=1$$

$$\int_{-a/2}^{a/2} dx\varphi*\varphi = |B|^2 \int_{-a/2}^{a/2} \cos^2\left(\frac{n\pi}{a}x\right)dx = |B|^2 \frac{1}{2}\int_{-a/2}^{a/2} 1-\cos\left(\frac{2n\pi}{a}x\right)dx = 1$$

$$|B|^2 \frac{1}{2}a = 1$$

$$B = \sqrt{\frac{2}{a}}$$

The solution looks like this:

$$<\varphi|\varphi>=1$$

$$\int_{-a/2}^{a/2} dx\varphi*\varphi = |B|^2 \int_{-a/2}^{a/2} \cos^2\left(\frac{n\pi}{a}x\right)dx = |B|^2 \frac{1}{2}\int_{-a/2}^{a/2} 1-\cos\left(\frac{2n\pi}{a}x\right)dx = 1$$

integral over cos is zero, integrate $\int_{-a/2}^{a/2} dx = a$

$$|B|^2 \frac{1}{2}a = 1$$

The energy eigenfunctions are:

$$\varphi_n = \sqrt{\frac{2}{a}}\cos\left(\frac{n\pi}{a}x\right) \qquad \text{where n} = 1,3,5,...$$

similarly, by normalizing to get A, as with (4.13) in book:

$$\varphi_n = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right) \qquad \text{where n} = 2,4,6,...$$

The energy eigenvalues are:

$$E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{2ma^2} n^2$$

4.11(b) Show that $\left(\hat{\mathbf{A}}\hat{\mathbf{B}}\right)^\dagger = \hat{\mathbf{B}}^\dagger\hat{\mathbf{A}}^\dagger$

Definition of hermitian adjoint $\hat{\mathbf{O}}^\dagger$ of $\hat{\mathbf{O}}$:

$$\left\langle \hat{\mathbf{O}}^\dagger\psi_\ell \,|\, \psi_n \right\rangle = \left\langle \psi_\ell \,|\, \hat{\mathbf{O}}\psi_n \right\rangle$$

Show that $\left\langle \left(\hat{\mathbf{A}}\hat{\mathbf{B}}\right)^\dagger \psi \,|\, \psi \right\rangle = \left\langle \hat{\mathbf{B}}^\dagger\hat{\mathbf{A}}^\dagger\psi \,|\, \psi \right\rangle$ to prove that $\left(\hat{\mathbf{A}}\hat{\mathbf{B}}\right)^\dagger = \hat{\mathbf{B}}^\dagger\hat{\mathbf{A}}^\dagger$

In Dirac Notation:

$$\begin{aligned}\left\langle \left(\hat{\mathbf{A}}\hat{\mathbf{B}}\right)^\dagger \psi \,|\, \psi \right\rangle &= \left\langle \psi \,|\, \left(\hat{\mathbf{A}}\hat{\mathbf{B}}\right)\psi \right\rangle \\ &= \left\langle \hat{\mathbf{A}}^\dagger\psi \,|\, \hat{\mathbf{B}}\psi \right\rangle \\ &= \left\langle \hat{\mathbf{B}}^\dagger\hat{\mathbf{A}}^\dagger\psi \,|\, \psi \right\rangle\end{aligned}$$

In Integral Notation:

$$\begin{aligned}\int_{-\infty}^{\infty} \left(\hat{\mathbf{A}}\hat{\mathbf{B}}\right)^\dagger \psi^*\psi\, dx &= \int_{-\infty}^{\infty} \psi^*\hat{\mathbf{A}}\hat{\mathbf{B}}\psi\, dx \\ &= \int_{-\infty}^{\infty} \hat{\mathbf{A}}^\dagger\psi^*\hat{\mathbf{B}}\psi\, dx \\ &= \int_{-\infty}^{\infty} \hat{\mathbf{B}}^\dagger\hat{\mathbf{A}}^\dagger\psi^*\psi\, dx\end{aligned}$$

4.14. If $\hat{\mathbf{A}}$ is Hermitian, show that $\langle \mathbf{A} \rangle$ is real; that is, show that $\langle \mathbf{A} \rangle^* = \langle \mathbf{A} \rangle$.

This proof uses, from the definition of inner products: $(AB)^* = B^*A^*$ and the property of Hermitian operators: If $\hat{\mathbf{A}}$ is Hermitian, then$\left(\hat{\mathbf{A}} = \hat{\mathbf{A}}^*\right)$.

$$\begin{aligned}\langle \mathbf{A} \rangle^* &= \left[\int_{-\infty}^{\infty} \psi^*\hat{\mathbf{A}}\psi\, dx\right]^* = \int_{-\infty}^{\infty} \left[\psi^*\left(\hat{\mathbf{A}}\psi\right)\right]^* dx = \int_{-\infty}^{\infty} (\hat{\mathbf{A}}\psi)^*\psi\, dx = \int_{-\infty}^{\infty} (\psi^*\hat{\mathbf{A}}^*)\psi\, dx \\ &= \int_{-\infty}^{\infty} (\psi^*\hat{\mathbf{A}})\psi\, dx \qquad \text{since } \hat{\mathbf{A}} = \hat{\mathbf{A}}^* \\ &= \langle \mathbf{A} \rangle\end{aligned}$$

Another proof: Start with the definition of a Hermitian operator:

$$\int_{-\infty}^{\infty} dx\phi_n^*(x)\hat{\mathbf{O}}\psi_m(x) = \left[\int_{-\infty}^{\infty} dx\psi_m^*(x)\hat{\mathbf{O}}\phi_n(x)\right]^*$$

$$\int_{-\infty}^{\infty} dx\phi_n^*(x)\hat{\mathbf{O}}\phi_n(x) = \left[\int_{-\infty}^{\infty} dx\phi_n^*(x)\hat{\mathbf{O}}\phi_n(x)\right]^*$$

$$\langle O_n \rangle = \left[\langle O_n \rangle\right]^*$$

4.17. Consider the operator $\hat{\mathbf{C}}$,

$$\hat{\mathbf{C}}\varphi(x) = \varphi^*(x)$$

(a) Is $\hat{\mathbf{C}}$ Hermitian?

This proof shows that the contrary assumption leads to contradiction. Start with definition of Hermitian operator. An operator is Hermitian if and only if:

$$\int_{-\infty}^{\infty} dx\phi_n^*(x)\hat{\mathbf{O}}\psi_m(x) = \left[\int_{-\infty}^{\infty} dx\psi_m^*(x)\hat{\mathbf{O}}\phi_n(x)\right]^*$$

$$\int_{-\infty}^{\infty} dx\phi_n^*(x)\hat{\mathbf{C}}\psi_m(x) = \int_{-\infty}^{\infty} dx\phi_n^*(x)\psi_m^*(x)$$

$$\left[\int_{-\infty}^{\infty} dx\psi_m^*(x)\hat{\mathbf{C}}\phi_n(x)\right]^* = \left[\int_{-\infty}^{\infty} dx\psi_m^*(x)\phi_n^*(x)\right]^* = \int_{-\infty}^{\infty} dx\phi_n(x)\psi_m(x)$$

since

$$\int_{-\infty}^{\infty} dx\phi_n^*(x)\psi_m^*(x) \neq \int_{-\infty}^{\infty} dx\phi_n(x)\psi_m(x)$$

$$\int_{-\infty}^{\infty} dx\phi_n^*(x)\hat{\mathbf{C}}\psi_m(x) \neq \left[\int_{-\infty}^{\infty} dx\psi_m^*(x)\hat{\mathbf{C}}\phi_n(x)\right]^*$$

$\therefore \hat{\mathbf{C}}$ is not Hermitian

(b) What are the eigenfunctions of $\hat{\mathbf{C}}$? (c) What are the eigenvalues of $\hat{\mathbf{C}}$?

$$\hat{\mathbf{C}}\varphi(x) = \varphi^*(x)$$

$$\hat{\mathbf{C}}\varphi(x) = c\varphi(x)$$

$$\therefore c\varphi(x) = \varphi^*(x)$$

$$\hat{\mathbf{C}}^2\varphi(x) = \hat{\mathbf{C}}\hat{\mathbf{C}}\varphi(x) = \hat{\mathbf{C}}\varphi^*(x) = \varphi(x)$$

$$\hat{\mathbf{C}}^2\varphi(x) = c^2\varphi(x)$$

$$\therefore c^2 = 1$$

$$c = \pm 1$$

since $c\varphi(x) = \varphi^*(x)$

if $\varphi(x) = \mathrm{Re}(\Psi)$, then $\mathrm{c} = 1$

if $\varphi(x) = \mathrm{Im}(\Psi)$, then $\mathrm{c} = i$ where Ψ = complex function in Hilbert Space.

(In Class) Prove that the eigenfunctions of a Hermitian operator are orthogonal to each other.

Show that the scalar product of two different eigenfunctions is zero.

Start with the definition of a Hermitian operator:

$$\int_{-\infty}^{\infty} dx \phi_n^*(x)\hat{\mathbf{O}}\psi_m(x) = \left[\int_{-\infty}^{\infty} dx \psi_m^*(x)\hat{\mathbf{O}}\phi_n(x)\right]^*$$

therefore

$$\int_{-\infty}^{\infty} dx \phi_n^*(x)\hat{\mathbf{O}}\phi_m(x) = \left[\int_{-\infty}^{\infty} dx \phi_m^*(x)\hat{\mathbf{O}}\phi_n(x)\right]^*$$

And the eigenvalue equations:

$$\hat{\mathbf{O}}\psi_n(x) = O_n\psi_n(x)$$

and

$$\hat{\mathbf{O}}\psi_m(x) = O_m\psi_m(x)$$

$$\int_{-\infty}^{\infty} dx \phi_m^*(x)\hat{\mathbf{O}}\phi_n(x) = \left[\int_{-\infty}^{\infty} dx \phi_n^*(x)\hat{\mathbf{O}}\phi_m(x)\right]^*$$

$$\int_{-\infty}^{\infty} dx \phi_m^*(x)\hat{\mathbf{O}}\phi_n(x) - \left[\int_{-\infty}^{\infty} dx \phi_n^*(x)\hat{\mathbf{O}}\phi_m(x)\right]^* = 0$$

$$\int_{-\infty}^{\infty} dx \phi_m^*(x)\hat{\mathbf{O}}\phi_n(x) - \int_{-\infty}^{\infty} dx \left(\hat{\mathbf{O}}\phi_m(x)\right)^* \phi_n(x) = 0$$

$$\int_{-\infty}^{\infty} dx \phi_m^*(x) O_n\phi_n(x) - \int_{-\infty}^{\infty} dx \left(O_m\phi_m(x)\right)^* \phi_n(x) = 0$$

$$\int_{-\infty}^{\infty} dx \phi_m^*(x) O_n\phi_n(x) - \int_{-\infty}^{\infty} dx \left(O_m\phi_m(x)\right)^* \phi_n(x) = 0$$

$$[O_n - O_m^*]\int_{-\infty}^{\infty} dx \phi_m^*(x)\phi_n(x) = 0$$

$$[O_n - O_m^*] \neq 0$$

$$\therefore \int_{-\infty}^{\infty} dx \phi_m^*(x)\phi_n(x) = 0$$

5.12. If $\hat{\mathbf{A}},\ \hat{\mathbf{B}},$ and $\hat{\mathbf{C}}$ are three distinct operators, show that:

(a) $\left[\hat{\mathbf{A}}+\hat{\mathbf{B}},\ \ \hat{\mathbf{C}}\right]=\left[\hat{\mathbf{A}},\ \ \hat{\mathbf{C}}\right]+\left[\hat{\mathbf{B}},\ \ \hat{\mathbf{C}}\right]$

$$\begin{aligned}\left[\hat{\mathbf{A}}+\hat{\mathbf{B}},\ \ \hat{\mathbf{C}}\right] &= \left(\hat{\mathbf{A}}+\hat{\mathbf{B}}\right)\hat{\mathbf{C}}-\hat{\mathbf{C}}\left(\hat{\mathbf{A}}+\hat{\mathbf{B}}\right)\\ &= \hat{\mathbf{A}}\hat{\mathbf{C}}+\hat{\mathbf{B}}\hat{\mathbf{C}}-\hat{\mathbf{C}}\hat{\mathbf{A}}-\hat{\mathbf{C}}\hat{\mathbf{B}}\\ &= \hat{\mathbf{A}}\hat{\mathbf{C}}-\hat{\mathbf{C}}\hat{\mathbf{A}}+\hat{\mathbf{B}}\hat{\mathbf{C}}-\hat{\mathbf{C}}\hat{\mathbf{B}}\\ &= \left[\hat{\mathbf{A}},\ \ \hat{\mathbf{C}}\right]+\left[\hat{\mathbf{B}},\ \ \hat{\mathbf{C}}\right]\end{aligned}$$

(b) $\left[\hat{\mathbf{A}}\hat{\mathbf{B}},\ \ \hat{\mathbf{C}}\right]=\hat{\mathbf{A}}\left[\hat{\mathbf{B}},\ \ \hat{\mathbf{C}}\right]+\left[\hat{\mathbf{A}},\ \ \hat{\mathbf{C}}\right]\hat{\mathbf{B}}$

$$\begin{aligned}\hat{\mathbf{A}}\left[\hat{\mathbf{B}},\ \ \hat{\mathbf{C}}\right]+\left[\hat{\mathbf{A}},\ \ \hat{\mathbf{C}}\right]\hat{\mathbf{B}} &= \hat{\mathbf{A}}(\hat{\mathbf{B}}\hat{\mathbf{C}}-\hat{\mathbf{C}}\hat{\mathbf{B}})+(\hat{\mathbf{A}}\hat{\mathbf{C}}-\hat{\mathbf{C}}\hat{\mathbf{A}})\hat{\mathbf{B}}\\ &= \hat{\mathbf{A}}\hat{\mathbf{B}}\hat{\mathbf{C}}-\hat{\mathbf{A}}\hat{\mathbf{C}}\hat{\mathbf{B}}+\hat{\mathbf{A}}\hat{\mathbf{C}}\hat{\mathbf{B}}-\hat{\mathbf{C}}\hat{\mathbf{A}}\hat{\mathbf{B}}\\ &= \hat{\mathbf{A}}\hat{\mathbf{B}}\hat{\mathbf{C}}-\hat{\mathbf{C}}\hat{\mathbf{A}}\hat{\mathbf{B}}\\ &= \left[\hat{\mathbf{A}}\hat{\mathbf{B}},\ \ \hat{\mathbf{C}}\right]\end{aligned}$$

5.13. If $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are both Hermitian, show that $\hat{\mathbf{A}}\hat{\mathbf{B}}$ is Hermitian if $\left[\hat{\mathbf{A}},\hat{\mathbf{B}}\right]=0.$

Using $\hat{\mathbf{A}}\hat{\mathbf{B}}=\hat{\mathbf{B}}\hat{\mathbf{A}}$ because they commute.

If $\hat{\mathbf{A}}^\dagger=\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}^\dagger=\hat{\mathbf{B}}$:

$$\left(\hat{\mathbf{A}}\hat{\mathbf{B}}\right)^\dagger=\hat{\mathbf{B}}^\dagger\hat{\mathbf{A}}^\dagger=\hat{\mathbf{B}}\hat{\mathbf{A}}$$

$\therefore$ If $\hat{\mathbf{A}}\hat{\mathbf{B}}=\hat{\mathbf{B}}\hat{\mathbf{A}}$

$$\left(\hat{\mathbf{A}}\hat{\mathbf{B}}\right)^\dagger=\hat{\mathbf{A}}\hat{\mathbf{B}}$$

$\hat{\mathbf{A}}\hat{\mathbf{B}}$ is Hermitian if $\hat{\mathbf{A}}\hat{\mathbf{B}}=\hat{\mathbf{B}}\hat{\mathbf{A}}$ which is true if $\left[\hat{\mathbf{A}},\hat{\mathbf{B}}\right]=0.$

6.16 (a) Show that $\hat{\wp}$ anticommutes with the momentum operator $\hat{p}$. This is, show that: $\left[\hat{\wp},\hat{p}\right]_+ \equiv \hat{\wp}\hat{p}+\hat{p}\hat{\wp}=0$

From the definition of the Parity operator: $\hat{\wp}\,\varphi(x)=\varphi(-x)$

Show that:

$$\hat{\wp}\hat{p}=-\hat{p}\hat{\wp}$$

$$\hat{\wp}\hat{p}=\hat{\wp}\left(-i\hbar\nabla\right)$$

$$-\hat{p}\hat{\wp}=i\hbar\nabla\,\hat{\wp}$$

$$\hat{\wp}\hat{p}\varphi_n(x)=\hat{\wp}\left(-i\hbar\frac{\partial}{\partial x}\right)\varphi_n(x)=\left(-i\hbar\frac{\partial}{\partial(-x)}\right)\hat{\wp}\,\varphi_n(x)=i\hbar\frac{\partial}{\partial x}\hat{\wp}\,\varphi_n(x)$$

$$-\hat{p}\hat{\wp}\,\varphi_n(x)=i\hbar\nabla\,\hat{\wp}\,\varphi_n(x)$$

$$\therefore\hat{\wp}\hat{p}\varphi_n(x)=-\hat{p}\hat{\wp}\,\varphi_n(x)$$

and

$$\hat{\wp}\hat{p}=-\hat{p}\hat{\wp}$$

(b) Use your answer to part (a) to show that $\hat{\wp}$ commutes with the kinetic energy operator $\hat{T}=\dfrac{\hat{p}^2}{2m}$.

The factor 1/2m is not important, just show that $\left[\hat{\wp},\hat{p}^2\right]=0$

$$\begin{aligned}\left[\hat{\wp},\hat{p}^2\right]&=\hat{\wp}\hat{p}\hat{p}-\hat{p}\hat{p}\hat{\wp}\\&=\hat{\wp}\hat{p}\hat{p}+\hat{p}(-\hat{p}\hat{\wp})\\&=\left(\hat{\wp}\hat{p}\right)\hat{p}+\hat{p}\hat{\wp}\hat{p}\\&=(-\hat{p}\hat{\wp})\hat{p}+\hat{p}\hat{\wp}\hat{p}\\&=-\hat{p}\hat{\wp}\hat{p}+\hat{p}\hat{\wp}\hat{p}\\&=0\end{aligned}$$

Prove the Cauchy-Schwartz Inequality
(from Dr. Shen's notes…)

$$0 \le \underbrace{\langle \varphi - \lambda\phi | \varphi - \lambda\phi \rangle}_{\text{vector length}} = \int (\varphi - \lambda\phi)^* (\varphi - \lambda\phi)\, dx$$

$$= \int (\varphi^* - \lambda^*\phi^*)(\varphi - \lambda\phi)\, dx$$

$$= \int (\varphi^*\varphi - \lambda^*\phi^*\varphi - \varphi^*\lambda\phi + \lambda^*\phi^*\lambda\phi) dx$$

$$= \int \varphi^*\varphi dx - \lambda^* \int \phi^*\varphi dx - \lambda \int \varphi^*\phi dx + |\lambda|^2 \int \phi^*\phi dx$$

$$= \langle \varphi|\varphi\rangle - \lambda^*\langle \phi|\varphi\rangle - \lambda\langle \varphi|\phi\rangle + |\lambda|^2 \langle \phi|\phi\rangle$$

Here is the trick: $\lambda = \frac{\langle \phi|\varphi\rangle}{\langle \phi|\phi\rangle}$ and $\lambda^* = \frac{\langle \varphi|\phi\rangle}{\langle \phi|\phi\rangle}$

Substitute:

$$= \langle \varphi|\varphi\rangle - \frac{\langle \varphi|\phi\rangle\langle \phi|\varphi\rangle}{\langle \phi|\phi\rangle} - \frac{\langle \phi|\varphi\rangle\langle \varphi|\phi\rangle}{\langle \phi|\phi\rangle} + \frac{\langle \phi|\varphi\rangle\langle \varphi|\phi\rangle}{\langle \phi|\phi\rangle \cancel{\langle \phi|\phi\rangle}} \cancel{\langle \phi|\phi\rangle}$$

$$0 \le \langle \varphi|\varphi\rangle - \frac{|\langle \varphi|\phi\rangle|^2}{\langle \phi|\phi\rangle} - \frac{|\langle \varphi|\phi\rangle|^2}{\langle \phi|\phi\rangle} + \frac{|\langle \varphi|\phi\rangle|^2}{\langle \phi|\phi\rangle}$$

$$0 \le \langle \varphi|\varphi\rangle - \frac{|\langle \varphi|\phi\rangle|^2}{\langle \phi|\phi\rangle} \cancel{- \frac{|\langle \varphi|\phi\rangle|^2}{\langle \phi|\phi\rangle} + \frac{|\langle \varphi|\phi\rangle|^2}{\langle \phi|\phi\rangle}}$$

$$0 \le \langle \varphi|\varphi\rangle\langle \phi|\phi\rangle - |\langle \varphi|\phi\rangle|^2$$

$$|\langle \varphi|\varphi\rangle|^2 \le \langle \varphi|\varphi\rangle\langle \phi|\phi\rangle$$

Prove the Robertson-Schrödinger relation:

$\hat{A}$ and $\hat{B}$ are Hermetian operators. $[\hat{A}, \hat{B}] = \hat{C}$ Prove that $(\Delta A)^2 (\Delta B)^2 \le \frac{1}{4}|\langle C\rangle|^2$

$$\langle A\psi | A\psi\rangle\langle B\psi | B\psi\rangle \ge |\langle B\psi | A\psi\rangle|^2$$

$$|\langle B\psi | A\psi\rangle|^2 \ge |im\langle B\psi | A\psi\rangle|^2$$

$$|im\langle B\psi | A\psi\rangle|^2 = \tfrac{1}{4}|2im\langle B\psi | A\psi\rangle|^2 = \tfrac{1}{4}|\langle B\psi | A\psi\rangle - \langle B\psi | A\psi\rangle *|^2$$

$$= \tfrac{1}{4}|\langle B\psi | A\psi\rangle - \langle A\psi | B\psi\rangle|^2 = \tfrac{1}{4}|\langle AB\psi | \psi\rangle - \langle BA\psi | \psi\rangle|^2$$

$$= \tfrac{1}{4}|\langle (AB - BA)\psi | \psi\rangle|^2 = \tfrac{1}{4}|\langle [A, B]\psi | \psi\rangle|^2 = \tfrac{1}{4}|\langle C\psi | \psi\rangle|^2$$

$$\therefore \langle A\psi | A\psi\rangle\langle B\psi | B\psi\rangle \le \tfrac{1}{4}|\langle C\psi | \psi\rangle|^2$$

$$\therefore |A|^2 |B|^2 \le \tfrac{1}{4}|\langle [A, B]\psi | \psi\rangle|^2$$

5.28 (p 143) The TDSE permits the identification $E = i\hbar\,\partial/\partial t$. Using this identification together with the rule (5.95), give a formal derivation of the uncertainty relation $\Delta E\Delta t \le \hbar/2$. Note that in the stationary state (eigenstate H), $\Delta E = 0$. The implication for this case is that a stationary state may last indefinitely.

From the Robertson-Schrodinger Equation:

$$(\Delta E)^2(\Delta t)^2 \le \tfrac{1}{4}\left|\langle[E,t]\rangle\right|^2$$

$$[E,t] = i\hbar\frac{\partial}{\partial t}t - ti\hbar\frac{\partial}{\partial t}$$

$$[E,t]g = i\hbar\frac{\partial}{\partial t}tg - ti\hbar\frac{\partial}{\partial t}g = i\hbar g + ti\hbar\frac{\partial}{\partial t}g - ti\hbar\frac{\partial}{\partial t}g = i\hbar g$$

$$[E,t] = i\hbar$$

$$\therefore |\Delta E|^2|\Delta t|^2 \le \tfrac{1}{4}|i\hbar|^2$$

$$\Delta E\Delta t \le \tfrac{\hbar}{2}$$

6.1.a.1 (page 165) Find $\psi(x,t)$ and $P(E_n)$ at $t>0$, relevant to a particle in a one-dimensional box with walls at (0,a) for each of the following states.

(1) $\psi(x,0) = A_1\sin(3\pi x/a)\cos(\pi x/a)$

Find normalization factor:

$$1 = \int_0^a|\psi(x,0)|^2\,dx = A_1^2\int_0^a\sin^2(3\pi x/a)\cos^2(\pi x/a)\,dx = A_1^2\,\frac{a}{4}$$ (from wolfram integrator)

$$A_1 = \frac{2}{\sqrt{a}}$$

Now find time dependence:

$$|\psi(x,t)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi(x,0)\rangle$$

$$|\psi(x,t)\rangle = \sum_n|\varphi_n\rangle e^{-\frac{i}{\hbar}Ht}\langle\varphi_n|\psi(x,0)\rangle$$

$$|\psi(x,0)\rangle = \sum_m c_m|\varphi_m\rangle$$

$$|\psi(x,t)\rangle = \sum_n|\varphi_n\rangle e^{-\frac{i}{\hbar}Ht}\left\langle\varphi_n\left|\sum_m c_m|\varphi_m\rangle\right.\right\rangle$$

$$|\psi(x,t)\rangle = \sum_n\sum_m c_m|\varphi_n\rangle e^{-\frac{i}{\hbar}Ht}\langle\varphi_n|\varphi_m\rangle = \sum_n\sum_m c_m|\varphi_n\rangle e^{-\frac{i}{\hbar}Ht}\delta_{nm} = \sum_n c_n e^{-\frac{i}{\hbar}E_nt}|\varphi_n\rangle$$

$$\langle\varphi_n|\psi(x,0)\rangle = \sum_m c_m\langle\varphi_n|\varphi_m\rangle = \sum_m c_m\delta_{nm} = c_n = \int dx\,\varphi_n^*\psi(x,0)$$

$$c_1 = \frac{4}{a}\int_0^a \sin^2(3\pi x/a)\cos^2(\pi x/a)\,dx = \frac{4}{a}\frac{a}{4} = 1$$

$$\left|\psi(x,t)\right\rangle = \frac{4}{a} e^{-iE_n t/\hbar} \sin(3\pi x/a)\cos(\pi x/a)$$

$$En = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$

6.10 (p 170) Show that $\frac{d}{dt}\langle A\rangle = 0$ in a stationary state, provided $\partial A/\partial t = 0$, using commutator relation (6.68).

$$\begin{aligned}\frac{d\langle A\rangle}{dt} &= \frac{i}{\hbar}\langle\varphi_n\left|[H,A]\varphi_n\right\rangle \\ &= \frac{i}{\hbar}\left(\langle\varphi_n|HA\varphi_n\rangle - \langle\varphi_n|AH\varphi_n\rangle\right) \\ &= \frac{i}{\hbar}\left(\langle H\varphi_n|A\varphi_n\rangle - \langle\varphi_n|AH\varphi_n\rangle\right) \\ &= \frac{i}{\hbar}E_n\left(\langle\varphi_n|A\varphi_n\rangle - \langle\varphi_n|A\varphi_n\rangle\right) = 0\end{aligned}$$

7.4 (page 198) The derivation in the text of the eigenvalues of $\hat{N}$ is based on the constraint that there are no states corresponding to the eigenvalues of n < -½. This constraint was guaranteed by setting $\hat{a}\varphi_0 = 0$. It would appear that it can also be guaranteed by setting $\hat{a}\varphi_{\frac{1}{2}} = 0$ for in the case:

$$\hat{a}\varphi_{\frac{1}{2}} = \varphi_{-\frac{1}{2}} = 0$$

Show that for $\varphi_{\frac{1}{2}}$ as defined is an eigenfunction of $\hat{N}$ with the eigenvalue zero; hence $\varphi_{\frac{1}{2}}$ is more properly termed φ_0.

$$\hat{a}\varphi_n = \varphi_{n-1}$$
$$\hat{a}\varphi_{\frac{1}{2}} = \varphi_{-\frac{1}{2}} = 0$$
$$\hat{N}\varphi_n = n\varphi_n$$
$$\hat{a}\dagger\hat{a}\varphi_n = n\varphi_n$$
$$\hat{a}\dagger\left(\hat{a}\varphi_{\frac{1}{2}}\right) = n\varphi_{\frac{1}{2}}$$
$$\hat{a}\dagger\left(\hat{0}\right) = n\varphi_{\frac{1}{2}} = 0$$
$$\therefore n = 0$$
$$\therefore \varphi_{\frac{1}{2}} = \varphi_0$$

7.5 (page 198) Using the fundamental commutator relation $\left[\hat{x}, \hat{p}\right] = i\hbar$ show that $\left[\hat{a}, \hat{a}\dagger\right] = 1$.

$$\left[\hat{a}, \hat{a}\dagger\right] = \left[\tfrac{1}{\sqrt{2}}\left(\beta\hat{x} + \tfrac{i}{\beta}\hat{p}\right), \tfrac{1}{\sqrt{2}}\left(\beta\hat{x} - \tfrac{i}{\beta}\hat{p}\right)\right] = \tfrac{1}{2}\left\{\left[\beta\hat{x}, -\tfrac{i}{\beta}\hat{p}\right] + \left[\tfrac{i}{\beta}\hat{p}, \beta\hat{x}\right]\right\} = \tfrac{1}{2}\left\{1+1\right\} = 1$$

7.8 (page 207) Show directly from the form of φ_n given by (7.57) $\varphi_n = A_n\left(\xi - \tfrac{\partial}{\partial\xi}\right)^n e^{-\xi^2/2}$ that $\hat{\wp}\,\varphi_n = (-1)^n\varphi_n$ where $\hat{\wp}$ is the parity operator.

$$\begin{aligned}
\hat{\wp}\,\varphi_n &= \varphi_n(-\xi) \\
&= A_n\left(-\xi - \tfrac{\partial}{\partial -\xi}\right)^n e^{-|-\xi|^2/2} \\
&= A_n\left(-\xi + \tfrac{\partial}{\partial\xi}\right)^n e^{-|-\xi|^2/2} \\
&= A_n(-1)^n\left(\xi - \tfrac{\partial}{\partial\xi}\right)^n e^{-\xi^2/2} \\
&= (-1)^n\varphi_n
\end{aligned}$$

7.9 (page 207) (a) Show that the normalized *n*th eigenstate φ_n is generated from the normalized ground state φ_0 through $\varphi_n = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n \varphi_0$.

$$\hat{a}^\dagger\left|\varphi_n\right\rangle = \sqrt{n+1}\left|\varphi_{n+1}\right\rangle$$
$$\left|\varphi_{n+1}\right\rangle = \left(n+1\right)^{-1/2}\hat{a}^\dagger\left|\varphi_n\right\rangle$$
$$\left|\varphi_{i+2}\right\rangle = \hat{a}^\dagger\left|\varphi_{i+1}\right\rangle = \left(i+2\right)^{-1/2}\left(i+1\right)^{-1/2}\hat{a}^\dagger\left|\varphi_i\right\rangle$$
$$\left|\varphi_{i+n}\right\rangle = \hat{a}^\dagger\left|\varphi_{i+n-1}\right\rangle = \left[\left(i+n\right)^{-1/2}\cdots\left(i+2\right)^{-1/2}\left(i+1\right)^{-1/2}\right]\hat{a}^\dagger\left|\varphi_i\right\rangle$$
$$i = 0$$
$$\left|\varphi_n\right\rangle = \hat{a}^\dagger\left|\varphi_{n-1}\right\rangle = \left[\left(n\right)^{-1/2}\cdots\left(2\right)^{-1/2}\left(1\right)^{-1/2}\right]\hat{a}^\dagger\left|\varphi_0\right\rangle$$
$$\left|\varphi_n\right\rangle = \left(n!\right)^{-1/2}\hat{a}^\dagger\left|\varphi_0\right\rangle$$

(b) Show that part (a) implies the following relations:

$$\varphi_n = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n \varphi_0$$
$$\sqrt{n!}\varphi_n = \left(\hat{a}^\dagger\right)^n \varphi_0$$
$$\hat{a}\sqrt{n!}\varphi_n = \hat{a}\left(\hat{a}^\dagger\right)^n \varphi_0$$
$$\sqrt{n!}\hat{a}\varphi_n = n\left(\hat{a}^\dagger\right)^{n-1} \varphi_0$$
$$\sqrt{n!}\hat{a}\varphi_n = n\sqrt{\left(n-1\right)!}\varphi_{n-1}$$
$$\sqrt{n}\hat{a}\varphi_n = n\varphi_{n-1}$$
$$\hat{a}\varphi_n = \sqrt{n}\varphi_{n-1}$$

7.10 (page 207) Show that the nth eigenstate of the harmonic oscillator, the average kinetic energy <T> is equal to the average potential energy <V> - the virial theorem. That is,

$$\langle V\rangle = \tfrac{k}{2}\langle x^2\rangle = \langle T\rangle = \tfrac{1}{2m}\langle p^2\rangle = \tfrac{1}{2}\langle E\rangle = \tfrac{\hbar\omega_0}{2}\left(n+\tfrac{1}{2}\right)$$

Using: $(a_+)|n\rangle = \sqrt{n+1}|n+1\rangle$ and $(a)|n\rangle = \sqrt{n}|n-1\rangle$

And:

$$\langle n|(aa_+)|n\rangle = \langle n|(a)\sqrt{n+1}|n+1\rangle = \langle n|\sqrt{n+1}\sqrt{n+1}|n+1-1\rangle = \langle n|n+1|n\rangle = n+1$$

$$\langle n|(a_+a)|n\rangle = \langle n|(a_+)\sqrt{n}|n-1\rangle = \langle n|\sqrt{n}\sqrt{n}|n-1+1\rangle = \langle n|n|n\rangle = n$$

$$\langle n|(a_+a_+)|n\rangle = \langle n|(a_+)\sqrt{n+1}|n+1\rangle = \langle n|\sqrt{n+2}\sqrt{n+1}|n+2\rangle = 0$$

$$\langle n|(aa)|n\rangle = \langle n|(a)\sqrt{n}|n-1\rangle = \langle n|\sqrt{n-1}\sqrt{n}|n-2\rangle = 0$$

For the Potential Energy: $x = \dfrac{a+a_+}{\sqrt{2}\beta}$ $\beta^2 = \dfrac{m\omega_0}{\hbar}$ $\omega_0^2 = \dfrac{k}{m}$

$$\begin{aligned}\langle V\rangle &= \langle n|\hat{V}|n\rangle \\ &= \langle n|\tfrac{k}{2}x^2|n\rangle \\ &= \tfrac{k}{4\beta^2}\langle n|(a+a_+)(a+a_+)|n\rangle \\ &= \tfrac{k}{4\beta^2}\langle n|\Big(\cancel{aa}^{\,0} + \cancel{aa_+}^{\,n+1} + \cancel{a_+a}^{\,n} + \cancel{a_+a_+}^{\,0}\Big)|n\rangle \\ &= \tfrac{k}{4\beta^2}(n+1+n) \\ &= \tfrac{k}{2\beta^2}\tfrac{\beta^2\hbar}{m\omega_0}\tfrac{m\omega_0^2}{k}(n+\tfrac{1}{2}) \\ &= \tfrac{\hbar\omega_0}{2}(n+\tfrac{1}{2})\end{aligned}$$

For the Kinetic Energy: $p = \dfrac{a-a_+}{\sqrt{2}\beta}\dfrac{m\omega_0}{i}$

$$\begin{aligned}\langle T\rangle &= \langle n|\hat{T}|n\rangle \\ &= \langle n|\tfrac{1}{2m}p^2|n\rangle \\ &= -\tfrac{m\omega_0^2}{4\beta^2}\langle n|(a-a_+)(a-a_+)|n\rangle \\ &= -\tfrac{m\omega_0^2}{4\beta^2}\langle n|\Big(\cancel{aa}^{\,0} - \cancel{aa_+}^{\,n+1} - \cancel{a_+a}^{\,n} + \cancel{a_+a_+}^{\,0}\Big)|n\rangle \\ &= \tfrac{m\omega_0^2}{4\beta^2}(n+1+n) \\ &= \tfrac{m\omega_0^2}{2\beta^2}\tfrac{\beta^2\hbar}{m\omega_0}(n+\tfrac{1}{2}) \\ &= \tfrac{\hbar\omega_0}{2}(n+\tfrac{1}{2})\end{aligned}$$

$$\therefore \langle V\rangle = \langle T\rangle$$

7.34 (page 221) Show that the current density J may be written $J = \frac{1}{2m}\left[\psi^* \hat{p}\psi + \left(\psi^* \hat{p}\psi\right)^*\right]$ where $\hat{p}$ is the momentum operator.

Show that it is equivalent to equation 7.107: $J = \frac{\hbar}{2mi}\left[\psi^* \nabla\psi - \psi\nabla\psi^*\right]$

Here is the math:

$$
\begin{aligned}
J &= \frac{\hbar}{2mi}\left[\psi^* \nabla\psi - \psi\nabla\psi^*\right] \\
&= \frac{\hbar}{2mi}\left[\psi^* \nabla\psi - \left(\psi^* \nabla^*\psi\right)^*\right] \\
&= \frac{1}{2m}\left[\psi^* \hat{p}\psi + \left(\psi^* \hat{p}\psi\right)^*\right] \\
&= \frac{1}{2m}\left[\psi^* \hat{p}\psi + \left(\psi^* \hat{p}\psi\right)^*\right] \\
&= \frac{1}{2m}\left[\psi^* \hat{p}\psi + \psi\hat{p}^*\psi^*\right] \\
&= \frac{1}{2m}\left[\psi^*\left(-i\hbar\nabla\right)\psi + \psi\left(-i\hbar\nabla\right)^*\psi^*\right] \\
&= \frac{1}{2m}\left[\psi^*\left(-i\hbar\nabla\right)\psi + \psi\left(i\hbar\nabla\right)\psi^*\right] \\
&= \frac{-i\hbar}{2m}\left[\psi^* \nabla\psi - \psi\nabla\psi^*\right] \\
&= \frac{\hbar}{2mi}\left[\psi^* \nabla\psi - \psi\nabla\psi^*\right] \quad \leftarrow \text{this is equation 7.107}
\end{aligned}
$$

7.35 (page 221) Show that for a one-dimensional wavefunction of the form [where $\phi(x,t)$ is real] $\psi(x,t) = A\exp\left[i\phi(x,t)\right]$, $\mathbf{J} = \frac{\hbar}{m}|A|^2\frac{\partial\phi}{\partial x}$.

Start with equation 7.107: $J = \frac{\hbar}{2mi}\left[\psi^* \nabla\psi - \psi\nabla\psi^*\right]$

$$
\begin{aligned}
J &= \frac{\hbar}{2mi}\left[\psi^* \nabla\psi - \psi\nabla\psi^*\right] \\
\psi(x,t) &= A\exp\left[i\phi(x,t)\right] \\
J &= \frac{\hbar}{2mi}\left[\psi^* \frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right] \\
&= \frac{\hbar}{2mi}\left[A^* e^{\left[-i\phi(x,t)\right]}\frac{\partial}{\partial x}Ae^{\left[i\phi(x,t)\right]} - Ae^{\left[i\phi(x,t)\right]}\frac{\partial}{\partial x}A^* e^{\left[-i\phi(x,t)\right]}\right] \\
&= \frac{\hbar}{2mi}|A|^2\left[e^{-i\phi}\frac{\partial}{\partial x}e^{i\phi} - e^{i\phi}\frac{\partial}{\partial x}e^{-i\phi}\right] \\
&= \frac{\hbar}{2mi}|A|^2\left[i\,\cancelto{1}{e^{-i\phi}e^{i\phi}}\,\frac{\partial\phi}{\partial x} + i\,\cancelto{1}{e^{i\phi}e^{-i\phi}}\,\frac{\partial\phi}{\partial x}\right] \\
&= \frac{\hbar}{2mi}|A|^2\,2i\frac{\partial\phi}{\partial x} \\
&= \frac{\hbar}{m}|A|^2\frac{\partial\phi}{\partial x}
\end{aligned}
$$

8.34 (page 323) Construct the eigenstates and eigenenergies of a particle in a two-dimensional rectangular box of edge lengths a and 2a. Take the origin to be at the corner of the rectangle. Account geometrically for the removal of most of the degeneracy present in the case of the square, two-dimensional box described previously. The degeneracy present for this configuration (e.g., the energy 5E is doubly degenerate) is sometimes call accidental degeneracy, in that it is neither exchange- nor symmetry-degenerate.

$$\varphi_{nxny}(x,y)=\varphi_{nx}(x)\varphi_{ny}(y)=\sqrt{\tfrac{2}{a}}\sin\tfrac{n_x\pi x}{a}\sqrt{\tfrac{2}{2a}}\sin\tfrac{n_y\pi y}{2a}$$

$$\varphi_{nx}(x)=\sqrt{\tfrac{2}{a}}\sin\tfrac{n_x\pi x}{a}$$

$$E_{nx}=n_x^2E_1$$

$$E_{ny}=\left(\tfrac{n_y}{2}\right)^2E_1$$

$$E_n=E_{nx}+E_{ny}=\left[n_x^2+\left(\tfrac{n_y}{2}\right)^2\right]E_1$$

Degeneracy – $n^2 = nx^2 + (ny/2)^2$

nx	nx^2	1 1	2 4	3 9	4 16	5 25	6 36	ny $(ny/2)^2$
1	1	2	5	10	17	26	37	
2	4	5	8	13	20	29	40	
3	9	10	13	18	25	34	45	
4	16	17	20	25	32	41	52	

For n=square_root(n^2) - Only survivor:

nx	nx^2	1 1	2 4	3 9	4 16	5 25	6 36	ny $(ny/2)^2$
1	1							
2	4							
3	9				5			
4	16			5				

Double degeneracy at 5E with nx,ny=(4,3) and (3,4).
Degeneracy occurs at Pythagorean triples.

8.35 (page 326) What is the order of degeneracy of the eigenstate $E_S=\hbar\omega_0(s+1)$ of the two-dimensional harmonic oscillator?

The degeneracy equals the number of ways of writing an integer s as the ordered sum of two whole integer numbers (starting at 0). There are (s+1) ways to do this.

8.36 (page 326) (a) Write down the Hamiltonians, eigenenergies, and eigenstates for a two-dimensional harmonic oscillator with distinct spring constants K_x and K_y.

$V(x)= K_x x^2/2=m\omega_x^2 x^2/2$ where $\omega_x^{2=}K_x/m$ and $V(y)= K_y y^2/2=m\omega_y^2 y^2/2$ where $\omega_y^{2=}K_y/m$

The Hamiltonian for this 2-D harmonic oscillator:

$$H=\frac{p_x^2}{2m}+\frac{K_x^2x^2}{2}+\frac{p_y^2}{2m}+\frac{K_y^2y^2}{2}=\frac{p_x^2}{2m}+\frac{m\omega_x^2x^2}{2}+\frac{p_y^2}{2m}+\frac{m\omega_y^2y^2}{2}=\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{m\omega_x^2x^2}{2}+\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial y^2}+\frac{m\omega_y^2y^2}{2}=H_x+H_y$$

The Schroedinger Equation is:

$$\frac{-\hbar^2}{2m}\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)+\frac{m\omega^2}{2}\left(x^2+y^2\right)\varphi=E\varphi$$

Separation of variables can be used to express φ as:

$$\varphi(x,y,z)=X(x)Y(y)$$

$$\frac{-\hbar^2}{2m}\left(\frac{\partial^2X(x)Y(y)}{\partial x^2}+\frac{\partial^2X(x)Y(y)}{\partial y^2}\right)+\left(\frac{m\omega_x^2x^2}{2}+\frac{m\omega_y^2y^2}{2}\right)X(x)Y(y)=EX(x)Y(y)$$

$$\frac{-\hbar^2}{2m}\frac{1}{X(x)Y(y)}\left(\frac{Y(y)\partial^2X(x)}{\partial x^2}+\frac{X(x)\partial^2Y(y)}{\partial y^2}\right)+\left(\frac{m\omega_x^2x^2}{2}+\frac{m\omega_y^2y^2}{2}\right)=E$$

$$\left(-\frac{\hbar^2}{2m}\frac{1}{X(x)}\frac{\partial^2X(x)}{\partial x^2}+\frac{m\omega_x^2x^2}{2}\right)+\left(-\frac{\hbar^2}{2m}\frac{1}{Y(y)}\frac{\partial^2Y(y)}{\partial y^2}+\frac{m\omega_y^2y^2}{2}\right)=E=E_x+E_y$$

Each variable is a 1-D Harmonic oscillator

$$\left(-\frac{\hbar^2}{2m}\frac{1}{X(x)}\frac{\partial^2X(x)}{\partial x^2}+\frac{m\omega_x^2x^2}{2}\right)=E_x$$

and

$$\left(-\frac{\hbar^2}{2m}\frac{1}{Y(y)}\frac{\partial^2Y(y)}{\partial y^2}+\frac{m\omega_y^2y^2}{2}\right)=E_y$$

Define:

$$\beta_x^2\equiv\frac{m\omega_x}{\hbar},\beta_y^2\equiv\frac{m\omega_y}{\hbar}$$

The eigenstates and eigenenergies of the Hamiltonians H_x and H_y are:

$$X_{n_x}(x)=A_{n_x}\mathcal{H}_{n_x}(\beta_xx)e^{-\beta_x^2x^2/2}\qquad E_{n_x}=\hbar\omega_x\left(n_x+\tfrac{1}{2}\right)=\hbar\sqrt{\frac{k_x}{m}}\left(n_x+\tfrac{1}{2}\right)\quad n_x=0,1,2,3,\cdots$$

$$Y_{n_y}(y)=A_{n_y}\mathcal{H}_{n_y}(\beta_yy)e^{-\beta_y^2y^2/2}\qquad E_{n_y}=\hbar\omega_y\left(n_y+\tfrac{1}{2}\right)=\hbar\sqrt{\frac{k_y}{m}}\left(n_x+\tfrac{1}{2}\right)\quad n_y=0,1,2,3,\cdots$$

Where $\mathcal{H}$ are the n^{th} order Hermite polynomials and A_n are the normalization constants.

The total eigenstate is and eigenenergy is:

$$\varphi_{n_xn_y}(x,y)=A_{n_xn_y}\mathcal{H}_{n_x}(\beta_xx)\mathcal{H}_{n_y}(\beta_yy)e^{-(\beta_x^2x^2+\beta_y^2y^2)/2}$$

$$E_{n_xn_y}=E_{n_x}+E_{n_y}=\hbar\omega_x\left(n_x+\tfrac{1}{2}\right)+\hbar\omega_y\left(n_y+\tfrac{1}{2}\right)$$

where

$$n=n_x+n_y\qquad n_x,n_y=0,1,2,3,\cdots$$

(b) If Ky=4Kx show that the eigenenergies may be written: $E_{n1n2} = \hbar\omega_0\left(n_1 + 2n_2 + \frac{3}{2}\right)$ where n_1 corresponds to x motion and n_y to y motion.

The eigenenergies of the Hamiltonians H_x and H_y are:

$$\omega_x = \sqrt{\tfrac{k_x}{m}} = \omega_0$$

$$\omega_y = \sqrt{\tfrac{k_y}{m}} = \sqrt{\tfrac{4k_x}{m}} = 2\sqrt{\tfrac{k_x}{m}} = 2\omega_0$$

$$E_{n_x} = \hbar\omega_x\left(n_x + \tfrac{1}{2}\right) == \hbar\omega_0\left(n_x + \tfrac{1}{2}\right) \quad n_x = 0,1,2,3,\cdots$$

$$E_{n_y} = \hbar\omega_y\left(n_y + \tfrac{1}{2}\right) = \hbar\omega_0\left(2n_y + 1\right) \quad n_y = 0,1,2,3,\cdots$$

$$E_{n_x n_y} = E_{n_x} + E_{n_y} = \hbar\omega_0\left(n_x + \tfrac{1}{2}\right) + \hbar\omega_0\left(2n_y + 1\right) = \hbar\omega_0\left(n_1 + 2n_2 + \tfrac{3}{2}\right)$$

where

$$n = n_1 + 2n_2 + 1 \qquad n_1, n_2 = 0,1,2,3,\cdots$$

(c) For part (b), what is the order of degeneracy of E2,3? List the corresponding eigenstates. Account for the absence of symmetry degeneracy among these states.

$$n = n_1 + 2n_2 + 1 \qquad n_1, n_2 = 0,1,2,3,\cdots$$

n=2+6+1=9

How many ways can we get n=9?

0 4
2 3
4 2
6 1
8 0

There are 5 different ways to get n=9. The absence of symmetry degeneracy arises because of the factor of 2 on n2.

Problem to solve. What is the ground state energy for each of the following 2-Particle systems?

1.) H_2, a deuteron and an electron
2.) He^+, a single ionized Helium atom
3.) Positronium, a bound positron and electron
4.) Exciton, with ε=10 (dielectric constant)

$$E_{0,1,0,0} = \cancel{\frac{\hbar^2 k^2}{2M}}^{\text{ignore CM system}} - \frac{Z'^2 \Re}{n^2}$$

$$\Re = \left(\frac{1}{4\pi\varepsilon}\right)^2 \frac{\mu e^4}{2\hbar^2} = \frac{\left(8.99\cdot 10^9 \text{ N m}^2\text{ C}^{-2}\right)^2 \left(9.109\cdot 10^{-31}\text{kg}\right)\left(1.6022\cdot 10^{-19}\text{C}\right)^4}{2\left(1.0546\cdot 10^{-34}\text{N m s}\right)^2\left(1.6022\cdot 10^{-19}\text{ J eV}^{-1}\right)} = 13.6 \text{ eV}$$

$m_e=9.109\cdot 10^{-31}$ kg
$m_p=1.6726\cdot 10^{-27}$ kg
$m_n=1.675\cdot 10^{-27}$ kg

	$Z' = \frac{Z}{\varepsilon}$	m1	m2	$\mu = \frac{m1\cdot m2}{m1+m2}$ kg	$\Re = \left(\frac{1}{4\pi\varepsilon}\right)^2 \frac{\mu e^4}{2\hbar^2}$ eV	$E = \frac{-Z'^2\Re}{n^2}$ eV
1	1	m_n+m_p	m_e	9.10652e-031	13.608698	-13.608698
2	2	$4m_p$	m_e	9.10652e-031	13.608698	-54.434782
3	1	m_e	m_e	4.5545e-03	6.806201	-6.806201
4	$\frac{1}{10}$	m_e	m_e	4.5545e-03	6.806201	-0.068062

9.5 p 365 Show that the frequencies of photons due to energy decays between successive levels of a rotator with moment of inertia *I* are given by $\hbar\omega = \frac{\hbar^2}{I}\left(l+1\right)$ or $\hbar\omega = \frac{\hbar^2}{I}\left(l\right)$.

$$\hat{H} = \frac{\hat{L}^2}{2I}$$

$$\hat{H}\left|n,l\right\rangle = \frac{\hat{L}^2}{2I}\left|n,l\right\rangle = \frac{1}{2I}\hbar^2 l(l+1)\left|n,l\right\rangle = E_{nl}\left|n,l\right\rangle$$

$$\hat{H}\left|n,l-1\right\rangle = \frac{\hat{L}^2}{2I}\left|n,l-1\right\rangle = \frac{1}{2I}\hbar^2 l(l-1)\left|n,l-1\right\rangle = E_{nl-1}\left|n,l-1\right\rangle$$

$$\hat{H}\left|n,l+1\right\rangle = \frac{\hat{L}^2}{2I}\left|n,l+1\right\rangle = \frac{1}{2I}\hbar^2 (l+1)(l+2)\left|n,l+1\right\rangle = E_{nl+1}\left|n,l+1\right\rangle$$

$$\Delta E = E_{nl} - E_{nl-1} = \frac{1}{2I}\hbar^2 l(l+1) - \frac{1}{2I}\hbar^2 l(l-1) = \frac{1}{2I}\hbar^2 2l = \frac{\hbar^2}{I}l$$

$$\Delta E = E_{nl+1} - E_{nl} = \frac{1}{2I}\hbar^2 (l+1)(l+2) - \frac{1}{2I}\hbar^2 l(l+1) = \frac{1}{2I}\hbar^2 2\left(l+1\right) = \frac{\hbar^2}{I}\left(l+1\right)$$

9.6 p 365 An HCl molecule may rotate as well as vibrate. Discuss the difference in emission frequencies associated with these two modes of excitation. Assume that only $l \to l \pm 1$ transitions between rotational states are allowed. Assume the same for vibrational levels. For rotational levels assume $l \,\square\!\leq 50$. Spring constant and moment of inertia may be inferred from the equivalent temperature values for HCl:
$\hbar\omega_0 / k_B = 4150K;\ \hbar^2 / 2Ik_B = 15.2K$

$$\hat{H}_{\text{rotational}} = \frac{\hat{L}^2}{2I}$$

$$\underbrace{\Delta E_{\text{rotational}}}_{l=1} = E_{nl} - E_{nl-1} = \frac{\hbar^2}{I}l \sim 2k_B(15.2K)l = \text{2(1)(15.2)(8.617 34·10-5K) eV/K=0.0026 eV}$$

$$\underbrace{\Delta E_{\text{rotational}}}_{l=50} = E_{nl} - E_{nl-1} = \frac{\hbar^2}{I}l \sim 2k_B(15.2K) = \text{2(50)(15.2)(8.617 34·10-5K) eV/K=0.13 eV}$$

$$E_{\text{vibrational}} = \hbar\omega_0\left(n+\tfrac{1}{2}\right)$$

$$\Delta E_{\text{vibrational}} = E_{nl+1} - E_{nl} = \hbar\omega_0\left(n+1+\tfrac{1}{2}\right) - \hbar\omega_0\left(n+\tfrac{1}{2}\right) = \hbar\omega_0 = 4150K \cdot \text{8.617 34·10-5eV/K=0.36eV}$$

The vibrational energy is orders of magnitude larger than the lowest rotational states, but around rotational states at *l*=50 it is only 3x energy.

9.23 (page 385) Assume that a particle has an orbital angular momentum with z component $\hbar m$ and square magnitude $\hbar^2 l\left(l+1\right)$.

(a) Show that in this state $\left\langle L_x \right\rangle = \left\langle L_y \right\rangle = 0$.

$$\left\langle L_x \right\rangle = \left\langle l,m \right| \hat{L}_x \left| l,m \right\rangle = \tfrac{1}{2}\left\langle l,m \right| \hat{L}_+ \left| l,m \right\rangle + \tfrac{1}{2}\left\langle l,m \right| \hat{L}_- \left| l,m \right\rangle$$
$$= \tfrac{1}{2}\hbar\sqrt{\left(l-m\right)\left(l+m+1\right)}\left\langle l,m \right| l,m+1 \rangle + \tfrac{1}{2}\hbar\sqrt{\left(l+m\right)\left(l-m+1\right)}\left\langle l,m \right| l,m-1 \rangle = 0$$
$$\left\langle L_y \right\rangle = \left\langle l,m \right| \hat{L}_y \left| l,m \right\rangle = \tfrac{-i}{2}\left\langle l,m \right| \hat{L}_+ \left| l,m \right\rangle - \tfrac{-i}{2}\left\langle l,m \right| \hat{L}_- \left| l,m \right\rangle$$
$$= \tfrac{-i}{2}\hbar\sqrt{\left(l-m\right)\left(l+m+1\right)}\left\langle l,m \right| l,m+1 \rangle - \tfrac{-i}{2}\hbar\sqrt{\left(l+m\right)\left(l-m+1\right)}\left\langle l,m \right| l,m-1 \rangle = 0$$

(b) Show that $\left\langle {L_x}^2 \right\rangle = \left\langle {L_y}^2 \right\rangle = 0$.

$$\hat{L}_x{}^2 = \tfrac{1}{2}\left(L_+ + L_-\right)\tfrac{1}{2}\left(L_+ + L_-\right) = \tfrac{1}{4}\left({L_+}^2 + L_-L_+ + L_+L_- + {L_-}^2\right)$$
$$\hat{L}_y{}^2 = \tfrac{-i}{2}\left(L_+ - L_-\right)\tfrac{-i}{2}\left(L_+ - L_-\right) = \tfrac{-1}{4}\left({L_+}^2 - L_-L_+ - L_+L_- + {L_-}^2\right) = \tfrac{1}{4}\left({L_+}^2 + L_-L_+ + L_+L_- + {L_-}^2\right)$$
$$\therefore \hat{L}_x{}^2 = \hat{L}_y{}^2 \Rightarrow \left\langle {L_x}^2 \right\rangle = \left\langle {L_y}^2 \right\rangle$$

$$\hat{L}^2 - \hat{L}_z^2 = \hat{L}_x^2 + \hat{L}_y^2 = 2\hat{L}_x^2 \Rightarrow \hat{L}_x^2 = \tfrac{1}{2}\left(\hat{L}^2 - \hat{L}_z^2\right)$$
$$\left\langle L_x^2 \right\rangle = \left\langle l,m \right| \hat{L}_x^2 \left| l,m \right\rangle = \left\langle l,m \right| \tfrac{1}{2}\left(\hat{L}^2 - \hat{L}_z^2\right)\left| l,m \right\rangle = \tfrac{1}{2}\left(\hbar^2 l\left(l+1\right) - m^2\hbar^2\right)\left\langle l,m \right| l,m \rangle$$

9.24 (p 385) The same conditions hold as in Problem 9.23. What is the expectation of the operator $\tfrac{1}{2}\left(L_xL_y + L_yL_x\right)$ in the Y_l^m?

$$\tfrac{1}{2}\left(L_xL_y + L_yL_x\right) = \tfrac{1}{2}\left[\tfrac{1}{2}\left(L_+ + L_-\right)\tfrac{-i}{2}\left(L_+ - L_-\right) + \tfrac{-i}{2}\left(L_+ - L_-\right)\tfrac{1}{2}\left(L_+ + L_-\right)\right]$$
$$= \tfrac{1}{2}\tfrac{1}{2}\tfrac{-i}{2}\left[\left(L_+L_+ - L_+L_- + L_-L_+ - L_-L_-\right) + \left(L_+L_+ + L_+L_- - L_-L_+ - L_-L_-\right)\right]$$
$$= \tfrac{1}{2}\tfrac{1}{2}\tfrac{-i}{2}2\left(L_+L_+ - L_-L_-\right) = \tfrac{-i}{4}\left({L_+}^2 - {L_-}^2\right)$$
$$\left\langle Y_l^m \right| \tfrac{1}{2}\left(L_xL_y + L_yL_x\right)\left| Y_l^m \right\rangle = \left\langle Y_l^m \right| \tfrac{-i}{4}\left({L_+}^2 - {L_-}^2\right)\left| Y_l^m \right\rangle$$
$$= \left\langle Y_l^m \right| \tfrac{-i}{4} L_+ c_{l,m+1} \left| Y_l^{m+1} \right\rangle - \left\langle Y_l^m \right| \tfrac{-i}{4} L_- d_{l,m-1} \left| Y_l^{m-1} \right\rangle$$
$$= \left\langle Y_l^m \right| \tfrac{-i}{4} c_{l,m+1}c_{l,m+2} \left| Y_l^{m+2} \right\rangle - \left\langle Y_l^m \right| \tfrac{-i}{4} d_{l,m-1}d_{l,m-2} \left| Y_l^{m-2} \right\rangle$$
$$= \tfrac{-i}{4} c_{l,m+1}c_{l,m+2} \left\langle Y_l^m \right| Y_l^{m+2} \rangle - \tfrac{-i}{4} d_{l,m-1}d_{l,m-2} \left\langle Y_l^m \right| Y_l^{m-2} \rangle = 0$$

10.6 (p 413) The current vector J associated with a wavefunction $\psi(\vec{r},t)$ is given by 7.107:

$$\vec{J} = \frac{\hbar}{2mi}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right).$$

The wavefunction may be termed source-free if $\nabla \cdot \vec{J} = 0$.

(a) What is the eigenfunction of $\hat{p}_r$ corresponding to the eigenvalue of $\hbar k$?

$$\hat{p}_r = -i\hbar \frac{1}{r}\frac{\partial}{\partial r} r$$

$$\hat{p}_r \varphi_k = \hbar k \varphi_k$$

$$-i\hbar \frac{1}{r}\frac{\partial}{\partial r} r\varphi_k = \hbar k \varphi_k$$

$$\frac{1}{r\varphi_k}\frac{\partial}{\partial r} r\varphi_k = ik$$

$$\varphi_k = A\frac{e^{ikr}}{r} \Rightarrow \varphi_k = A\frac{e^{ikr}}{r} e^{-iEt/\hbar} = \frac{A}{r} e^{i(kr-\omega t)}$$

(b) Calculate $\nabla \cdot \vec{J}$ for this eigenfunction.

$$\vec{J} = \frac{\hbar}{2mi}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right) = \frac{\hbar}{2mi}\left(\frac{A}{r}e^{-i(kr-\omega t)}\nabla \frac{A}{r}e^{i(kr-\omega t)} - \frac{A}{r}e^{i(kr-\omega t)}\nabla \frac{A}{r}e^{-i(kr-\omega t)}\right)$$

$$\nabla \frac{A}{r}e^{i(kr-\omega t)} = \left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right)\frac{A}{r}e^{i(kr-\omega t)} = \left(\frac{A}{r^2}\frac{\partial}{\partial r}\left(-1+ikr\right)e^{i(kr-\omega t)}\right)$$

$$= \left(\frac{A}{r^2}(ik)e^{i(kr-\omega t)} + (-1+ikr)ike^{i(kr-\omega t)}\right) = \left(\frac{Aik}{r}e^{i(kr-\omega t)}\right)$$

$$\nabla \frac{A}{r}e^{-i(kr-\omega t)} = \left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right)\frac{A}{r}e^{-i(kr-\omega t)} = \left(\frac{A}{r^2}\frac{\partial}{\partial r}\left(-1-ikr\right)e^{-i(kr-\omega t)}\right)$$

$$= \left(\frac{A}{r^2}(ik)e^{-i(kr-\omega t)} + (-1-ikr)ike^{-i(kr-\omega t)}\right) = \left(\frac{-Aik}{r}e^{-i(kr-\omega t)}\right)$$

$$\vec{J} = \frac{\hbar}{2mi}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right) = \frac{\hbar}{2mi}\left(\frac{A}{r}e^{-i(kr-\omega t)}\frac{Aik}{r}e^{i(kr-\omega t)} - \frac{A}{r}e^{i(kr-\omega t)}\frac{-Aik}{r}e^{-i(kr-\omega t)}\right)$$

$$= \frac{k\hbar}{m}\frac{A^2}{r^2}$$

11.45 p 515 (a) For spin corresponding to s=1/2, show that the eigenvectors S_x and S_y are:

$$\alpha_x = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \quad \alpha_y = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix} \quad \beta_x = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \quad \beta_y = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$$

Obeys the eigenvalues equations for s=1/2.

$$S_z\left(\alpha_x+\beta_x\right) = \frac{\hbar}{2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}+\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}\right) = \frac{\hbar}{2\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}+\frac{\hbar}{2\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{\hbar}{2}\left(\beta_x+\alpha_x\right)$$

$$S_z\left(\alpha_y+\beta_y\right) = \frac{\hbar}{2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}+\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}\right) = \frac{\hbar}{2\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}+\frac{\hbar}{2\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix} = \frac{\hbar}{2}\left(\beta_y+\alpha_y\right)$$

(b) What are the eigenvalues corresponding to these eigenvectors?

$$S_x\alpha_x = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{\hbar}{2\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{\hbar}{2}\alpha_x$$

$$S_x\beta_x = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{\hbar}{2\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{\hbar}{2}\beta_x$$

$$S_y\alpha_y = \frac{i\hbar}{2}\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix} = \frac{i\hbar}{2\sqrt{2}}\begin{bmatrix}-i\\1\end{bmatrix} = \frac{\hbar}{2\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix} = \frac{\hbar}{2}\alpha_y$$

$$S_y\beta_y = \frac{i\hbar}{2}\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix} = \frac{i\hbar}{2}\frac{1}{\sqrt{2}}\begin{bmatrix}i\\1\end{bmatrix} = \frac{\hbar}{2\sqrt{2}}\begin{bmatrix}-1\\i\end{bmatrix} = \frac{-\hbar}{2\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix} = \frac{-\hbar}{2}\beta_y$$

(c) Show that the eigenvectors comprise two sets of orthonormal vectors.

$$\langle\alpha_x|\alpha_x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix} = \frac{1}{2}(1+1) = 1 \qquad \langle\alpha_y|\alpha_y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\end{bmatrix} = \frac{1}{2}(1+1) = 1$$

$$\langle\alpha_x|\beta_x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix} = \frac{1}{2}(1-1) = 0 \qquad \langle\alpha_y|\beta_y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\end{bmatrix} = \frac{1}{2}(1-1) = 0$$

$$\langle\beta_x|\alpha_x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix} = \frac{1}{2}(1-1) = 0 \qquad \langle\beta_y|\alpha_y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\end{bmatrix} = \frac{1}{2}(1-1) = 0$$

$$\langle\beta_x|\beta_x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix} = \frac{1}{2}(1+1) = 1 \qquad \langle\beta_y|\beta_y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\end{bmatrix} = \frac{1}{2}(1+1) = 1$$