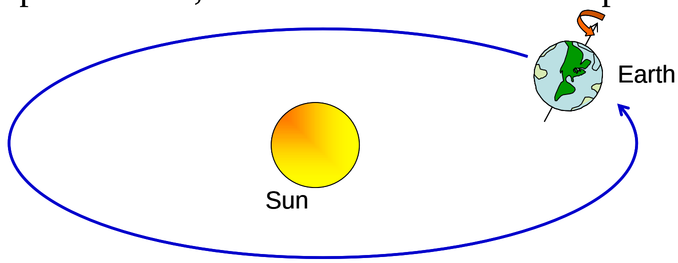
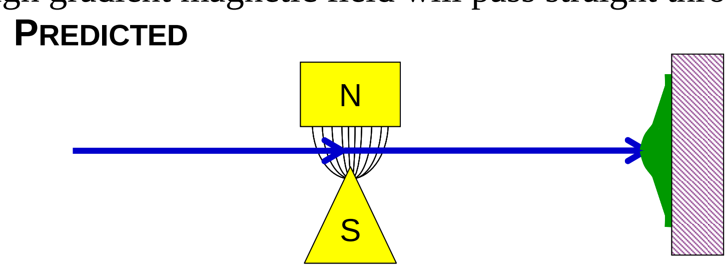
Spin and the Pauli Spin Matrices (Section 11.6 p 152)

Today we are going to study spin. You know that the Earth spins as it goes around the Sun. Spin results in day and night. An electron also has rotation and two states, up and down, for the orientation of the spin.

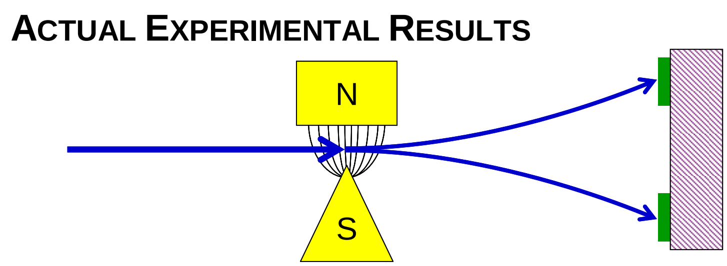


Stern-Gerlach Experiment

(see page 524). From the classical point of view, an electron beam passing through a high gradient magnetic field will pass straight through.



In a constant magnetic field, a dipole experiences no net force. A change in magnetic field exerts a force on a dipole. The experimental results could be described by electron spin creating a dipole – up or down – splitting and deflecting the beam.



Analogy to Angular Momentum Theory

We have already learned about the $\hat{L}^2$, $\hat{L}_z$. Spin is a vector operator that has the same form: $\vec{S}$. We have the same commutator relations:

$$\left[S_x, S_y\right] = i\hbar S_z$$

$$\left[S_y, S_z\right] = i\hbar S_x$$

$$\left[S_z, S_x\right] = i\hbar S_y$$

also we have

$$S_{\pm} = S_x \pm iS_y$$
$$S^2 \left| s, m_S \right\rangle = \hbar^2 s(s+1) \left| s, m_S \right\rangle$$
$$S_z \left| s, m_S \right\rangle = \hbar m_S \left| s, m_S \right\rangle$$

So we have the same results as angular momentum but this is called spin. For example, for an electron, the spin s=½, is a half integral number.

Here is the Dirac notation for an eigenfunction α of the eigenstate with s=½, m_s=½.

$$\alpha = \left| s = ½, m_S = ½ \right\rangle$$

Consider the S_Z and S^2 operators operating on these eigenstates:

$$\alpha = \left| s = ½, m_S = ½ \right\rangle$$
$$\beta = \left| s = ½, m_S = -½ \right\rangle$$
$$S_z \alpha = S_z \left| s = ½, m_S = ½ \right\rangle = ½\hbar \left| s = ½, m_S = ½ \right\rangle = ½\hbar\alpha$$
$$S_z \beta = -½\hbar\beta$$
$$S^2 \alpha = S^2 \left| s = ½, m_S = ½ \right\rangle = ½(1+ ½)\hbar \left| s = ½, m_S = ½ \right\rangle = ¾\hbar^2\alpha$$
$$S^2 \beta = ¾\hbar^2\beta$$

Spin up, spin down, the solutions describe the electron – it has 2D space. Now we are only considering the spin state.

$$\psi = c_1\alpha + c_2\beta$$

S_x, S_y, and S_z are all operators. If you write out operators, this is the basis. Write out a matrix (each Hii also has an eigenvalue and eigenfunction).

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$
$$H_{ij}^* = H_{ji}$$

In matrix form,

$$[S_x] = \begin{bmatrix} \langle \alpha | S_x | \alpha \rangle & \langle \alpha | S_x | \beta \rangle \\ \langle \beta | S_x | \alpha \rangle & \langle \beta | S_x | \beta \rangle \end{bmatrix}$$

operates on wavefunction $= \begin{bmatrix} a \\ b \end{bmatrix}$

The ladder operator is exactly the same as in the angular momentum theory:

$$S_+ |s, m_S\rangle = \hbar\sqrt{s(s+1) - m_S(m_S+1)}\,|s, m_S+1\rangle$$
$$S_- |s, m_S\rangle = \hbar\sqrt{s(s+1) - m_S(m_S-1)}\,|s, m_S-1\rangle$$
$$S_+\alpha = 0 \qquad S_+\beta = \hbar\alpha$$
$$S_-\alpha = \hbar\beta \qquad S_-\beta = 0$$

Let's write it out:

$$S_+\alpha = S_+|s = ½, m_S = ½\rangle = \hbar\sqrt{½(½+1) - ½(½+1)}\,|s = ½, m_S = ½+1\rangle = 0$$
$$S_+\beta = S_+|s = ½, m_S = -½\rangle = \hbar\sqrt{½(½+1) - ½(-½+1)}\,|s = ½, m_S = -½+1\rangle = \hbar|\alpha\rangle$$
$$S_-\alpha = S_-|s = ½, m_S = ½\rangle = \hbar\sqrt{½(½+1) + ½(½-1)}\,|s = ½, m_S = ½-1\rangle = \hbar|\beta\rangle$$
$$S_-\beta = S_-|s = ½, m_S = -½\rangle = \hbar\sqrt{½(½+1) + ½(-½-1)}\,|s = ½, m_S = -½-1\rangle = 0$$

We have:

$$\alpha = |s = ½, m_S = ½\rangle$$
$$\beta = |s = ½, m_S = -½\rangle$$
$$S_z\alpha = ½\hbar\alpha$$
$$S_z\beta = -½\hbar\beta$$

Recall earlier:

$$S_{\pm} = S_x \pm iS_y$$

You will write it out:

① $S_+\alpha = S_x + iS_y\left|\alpha\right\rangle = 0$

② $S_-\alpha = S_x - iS_y\left|\alpha\right\rangle = \hbar\left|\beta\right\rangle$

③ $S_+\beta = S_x + iS_y\left|\beta\right\rangle = \hbar\left|\alpha\right\rangle$

④ $S_-\beta = S_x - iS_y\left|\beta\right\rangle = 0$

Adding and subtracting equations 1 and 2 results in:

$$S_x\left|\alpha\right\rangle = \frac{\hbar}{2}\left|\beta\right\rangle$$

$$S_y\left|\alpha\right\rangle = \frac{i\hbar}{2}\left|\beta\right\rangle$$

Adding and subtracting equations 3 and 4 results in:

$$S_x\left|\beta\right\rangle = \frac{\hbar}{2}\left|\alpha\right\rangle$$

$$S_y\left|\beta\right\rangle = -\frac{i\hbar}{2}\left|\alpha\right\rangle$$

Then for example with a matrix you get:

$$\left\langle\alpha\middle|S_x\middle|\alpha\right\rangle = \left\langle\alpha\middle|\beta\right\rangle\tfrac{\hbar}{2} = 0$$

$$\left\langle\alpha\middle|S_x\middle|\beta\right\rangle = \left\langle\alpha\middle|\alpha\right\rangle\tfrac{\hbar}{2} = \tfrac{\hbar}{2}$$

$$\left\langle\beta\middle|S_x\middle|\beta\right\rangle = \left\langle\beta\middle|\alpha\right\rangle\tfrac{\hbar}{2} = \tfrac{\hbar}{2}$$

$$\left\langle\beta\middle|S_x\middle|\alpha\right\rangle = \left\langle\beta\middle|\beta\right\rangle\tfrac{\hbar}{2} = 0$$

$$\left[S_x\right] = \begin{bmatrix}\left\langle\alpha\middle|S_x\middle|\alpha\right\rangle & \left\langle\alpha\middle|S_x\middle|\beta\right\rangle \\ \left\langle\beta\middle|S_x\middle|\alpha\right\rangle & \left\langle\beta\middle|S_x\middle|\beta\right\rangle\end{bmatrix} = \begin{bmatrix}0 & \tfrac{\hbar}{2} \\ \tfrac{\hbar}{2} & 0\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}$$

$$\left[S_y\right] = \begin{bmatrix}\left\langle\alpha\middle|S_y\middle|\alpha\right\rangle & \left\langle\alpha\middle|S_y\middle|\beta\right\rangle \\ \left\langle\beta\middle|S_y\middle|\alpha\right\rangle & \left\langle\beta\middle|S_y\middle|\beta\right\rangle\end{bmatrix} = \begin{bmatrix}\left\langle\alpha\middle|\beta\right\rangle\tfrac{i\hbar}{2} & \left\langle\alpha\middle|\alpha\right\rangle\tfrac{-i\hbar}{2} \\ \left\langle\beta\middle|\beta\right\rangle\tfrac{i\hbar}{2} & \left\langle\beta\middle|\alpha\right\rangle\tfrac{-i\hbar}{2}\end{bmatrix} = \frac{i\hbar}{2}\begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}$$

$$\left[S_z\right] = \begin{bmatrix}\left\langle\alpha\middle|S_z\middle|\alpha\right\rangle & \left\langle\alpha\middle|S_z\middle|\beta\right\rangle \\ \left\langle\beta\middle|S_z\middle|\alpha\right\rangle & \left\langle\beta\middle|S_z\middle|\beta\right\rangle\end{bmatrix} = \begin{bmatrix}\left\langle\alpha\middle|\alpha\right\rangle\tfrac{\hbar}{2} & \left\langle\alpha\middle|\beta\right\rangle\tfrac{\hbar}{2} \\ \left\langle\beta\middle|\alpha\right\rangle\tfrac{\hbar}{2} & \left\langle\beta\middle|\beta\right\rangle\tfrac{\hbar}{2}\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$$

Final Results:

$$[S_x] = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad [S_y] = \frac{i\hbar}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad [S_z] = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underbrace{\alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{Spin Up}} \qquad \underbrace{\beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{Spin Down}}$$

Pauli developed this theory so we call it the Pauli matrix. The Pauli operator is defined by (page 515):

$$\hat{\boldsymbol{\sigma}} = \frac{2}{\hbar}\hat{\mathbf{S}}$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y = i\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example – we have spin corresponding to s=1/2, what is the S_x eigenvalue of the eigenvector $\alpha_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

$$S_x\alpha_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{\hbar}{2\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{\hbar}{2}\alpha_x$$

Homework: 11.45