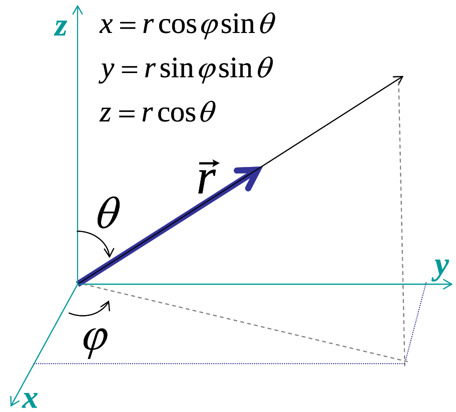
Today, we consider the angular momentum operators in different coordinate systems. [Note: follows Section 9.3, page 367]



The eigenfunctions of L^2 and L_z

$$\hat{L}^2\varphi_{lm} = \hbar^2 l(l+1)\varphi_{lm}$$
$$\hat{L}_z\varphi_{lm} = \hbar m\varphi_{lm}$$

You may remember we want to express the details of the angular momentum.

$$\hat{L}_x = -i\hbar\left(y\tfrac{\partial}{\partial z} - z\tfrac{\partial}{\partial y}\right)$$
$$\hat{L}_y = -i\hbar\left(z\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial z}\right)$$
$$\hat{L}_z = -i\hbar\left(x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x}\right)$$

The chain is very complicated. Remember this from math.

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$$

For example, we calculate L_z=[ ] using above relations. We first have to calculate:

$$\frac{\partial r}{\partial x} \quad \frac{\partial \theta}{\partial x} \quad \frac{\partial \phi}{\partial x} \quad \frac{\partial r}{\partial y} \quad \frac{\partial \theta}{\partial y} \quad \frac{\partial \phi}{\partial y}$$

We use these relations:

$$r^2 = x^2 + y^2 + z^2 \quad \cos\theta = \tfrac{y}{r} \quad \tan\phi = \tfrac{y}{x}$$

Here is an outline of the math involved (see problem 9.14, page 381):

① $$\frac{\partial r}{\partial x} = \frac{2x}{2}\left(x^2 + y^2 + z^2\right)^{-1/2} = \frac{x}{r}$$

② $$-\frac{\sin\theta\,\partial\theta}{\partial x} = -\frac{y}{r^2}\frac{\partial r}{\partial x} \quad \Rightarrow \quad \frac{\partial\theta}{\partial x} = \frac{\cos\phi\cos\theta}{r}$$

③ $$\frac{\partial\phi}{\partial x} = -\frac{y\cos^2\phi}{x^2}$$

④ $$\frac{\partial r}{\partial x} = \frac{y}{r}$$

⑤ $$\frac{\partial\theta}{\partial y} = \frac{\sin\phi\cos\theta}{r}$$

⑥ $$\frac{\partial\phi}{\partial y} = \frac{\cos^2\phi}{x}$$

These 6 relations put into here and calculate.

$$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi} = \frac{y}{r}\frac{\partial}{\partial r} + \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{\cos^2\phi}{x}\frac{\partial}{\partial\phi}$$

$$x\frac{\partial}{\partial y} = \frac{xy}{r}\frac{\partial}{\partial r} + \frac{x\sin\phi\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{x\cos^2\phi}{x}\frac{\partial}{\partial\phi} = \frac{xy}{r}\frac{\partial}{\partial r} + \frac{x\sin\phi\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{x\cos^2\phi}{x}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi} = \frac{x}{r}\frac{\partial}{\partial r} + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta} - \frac{y\cos^2\phi}{x^2}\frac{\partial}{\partial\phi}$$

$$y\frac{\partial}{\partial x} = \frac{yx}{r}\frac{\partial}{\partial r} + \frac{y\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta} - \frac{y^2\cos^2\phi}{x^2}\frac{\partial}{\partial\phi}$$

$$\begin{aligned}
\hat{L}_z &= -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \\
&= -i\hbar\left(\cancel{\frac{xy}{r}\frac{\partial}{\partial r}} + \frac{x\sin\phi\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{x\cos^2\phi}{x}\frac{\partial}{\partial\phi} \cancel{- \frac{yx}{r}\frac{\partial}{\partial r}} - \frac{y\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{y^2\cos^2\phi}{x^2}\frac{\partial}{\partial\phi}\right) \\
&= -i\hbar\left(\left[\frac{x\sin\phi\cos\theta}{r}\frac{y}{r\sin\theta\sin\phi} - \frac{y\cos\phi\cos\theta}{r}\frac{x}{r\sin\theta\cos\phi}\right]\frac{\partial}{\partial\theta} + \left[\frac{x\cos^2\phi}{x} + \frac{y^2\cos^2\phi}{x^2}\right]\frac{\partial}{\partial\phi}\right) \\
&= -i\hbar\left(\cancel{\left[\frac{xy}{r^2\tan\theta} - \frac{xy}{r^2\tan\theta}\right]}\frac{\partial}{\partial\theta} + \left[1 + \tan^2\phi\right]\cos^2\phi\frac{\partial}{\partial\phi}\right) \\
&= -i\hbar\left(\left[\cos^2\phi + \sin^2\phi\right]\frac{\partial}{\partial\phi}\right) \\
&= -i\hbar\left(\frac{\partial}{\partial\phi}\right)
\end{aligned}$$

$$\boxed{\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}}$$ very simple relation for the z component.

If you have time, you can calculate these for yourself:

$$\hat{L}_x = i\hbar\left(\sin\phi \frac{\partial}{\partial\theta} + \cos\theta\cos\phi \frac{\partial}{\partial\phi}\right)$$
$$\hat{L}_y = i\hbar\left(-\cos\phi \frac{\partial}{\partial\theta} + \cot\theta\sin\phi \frac{\partial}{\partial\phi}\right)$$
$$\hat{L}_z = -i\hbar \frac{\partial}{\partial\phi}$$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{1}\frac{\partial}{\partial\phi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Similar to momentum operator.

$$\hat{L}_z\psi_{lm} = -i\hbar\frac{\partial}{\partial\phi}\psi_{lm} = m\hbar\psi_{lm}$$
$$\phi_{lm} = \phi_l\phi_m = \phi_l e^{im\phi}$$
$$\psi_m(\phi) = \frac{1}{\sqrt{2}} e^{im\phi}$$
$$\int_0^{2\pi}\psi_m^*(\phi)\psi(\phi)\,d\phi = \int_0^{2\pi}\frac{d\theta}{2\pi} = 1$$

and the total wavefunction

$$\phi_{lm} = \frac{1}{\sqrt{2\pi}} e^{im\phi} \underbrace{\Theta_l^m(\theta)}_{\substack{\textit{amplitude} \\ \text{this is also an} \\ \text{eigenfunction} \\ \text{of } \phi_{lm}}}$$

Now we have this equation:

$$\hat{L}^2\phi_{lm} = \hbar^2 l(l+1)\phi_{lm}$$
$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{1}\frac{\partial\Theta}{\partial\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$$

consider $m = 0$

$\Theta_l = P_l(\theta)$ these are the Legendre Polynomial Functions

from this we get

$$P_l^m = (-1)^m(1-\mu)^{m/2}\frac{\partial^m}{\partial\mu^m}P_l(\mu) \quad \text{where} \quad \mu \equiv \cos\theta$$

$$\phi_{lm} = Y_l^m(\theta,\phi) = \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi}$$

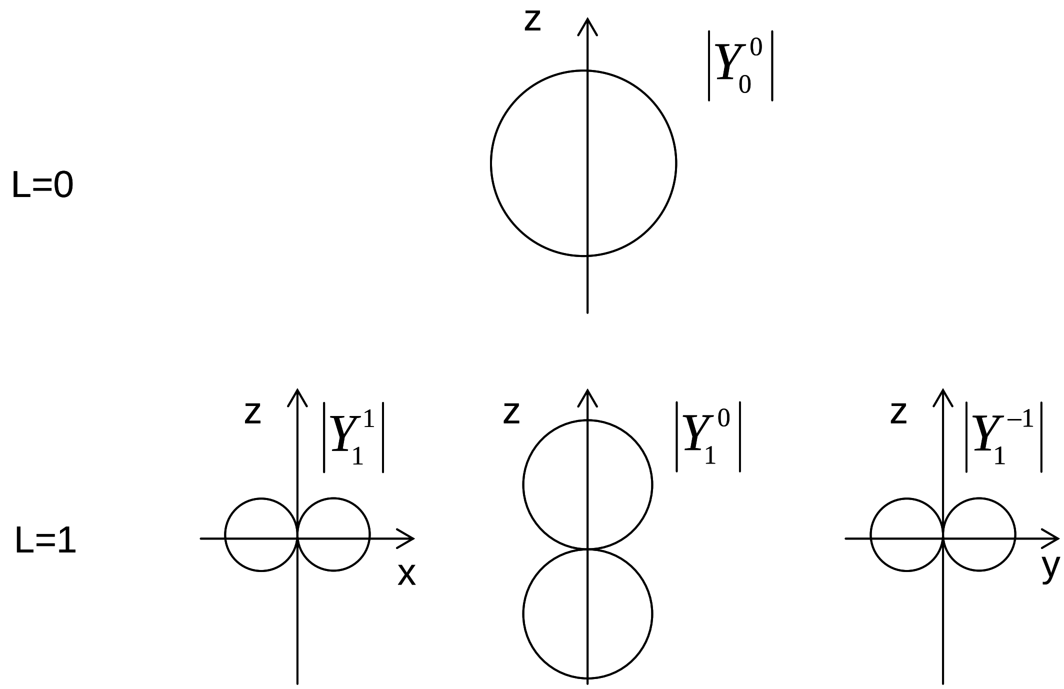
Let's look at these eigenfunctions (see Table 9.1 and Figure 9.2 starting at page 374):

$$l=0 \qquad P_0 = 1 \qquad Y_0^0 = \left(\tfrac{1}{4\pi}\right)^{1/2}$$

$$l=1 \qquad P_1^1 = -\sin\theta \qquad Y_1^1 = -\tfrac{1}{2}\left(\tfrac{3}{2\pi}\right)^{1/2}\sin\theta e^{i\phi}$$

$$P_1^0 = \cos\theta \qquad Y_1^0 = \tfrac{1}{2}\left(\tfrac{3}{\pi}\right)^{1/2}\cos\theta$$

$$P_1^{-1} = \tfrac{1}{2}\sin\theta \qquad Y_1^{-1} = \tfrac{1}{2}\left(\tfrac{3}{2\pi}\right)^{1/2}\sin\theta e^{-i\phi}$$



Very simple to do Dirac notation – based on real notation.

$Y_l^m \Rightarrow |l,m\rangle$

$L^2|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle$

$L_z|l,m\rangle = \hbar m|l,m\rangle \qquad m = -l, -l+1, ..., 0, ..., l-1, l$

$L_+|l,m\rangle = \hbar\left[(l-m)(l+m+1)\right]^{1/2}|l,m+1\rangle$

$L_-|l,m\rangle = \hbar\left[(l+m)(l-m+1)\right]^{1/2}|l,m-1\rangle$

Y_l^m is a normalized eigenfunction

$\langle l',m'|l,m\rangle = \delta_{ll'}\delta_{mm'}$

$L_x = \frac{1}{2}\left[L_+ + L_-\right]$

$L_y = \frac{i}{2}\left[L_- - L_+\right]$

to calculate the average - very simple to get directly if you use Ladder operator

$\langle l,m|L_x|l,m\rangle = \langle l,m|L_y|l,m\rangle = 0$

Hint: this is homework.

$\langle l,m|L_x^2|l,m\rangle = ?$

$\langle l,m|L^2|l,m\rangle = \frac{1}{2}\left[\hbar^2 l(l+1) - m^2\hbar^2\right]$

Today's homework: 9.23, 9.24 - use Dirac notation, they are easy to solve.