Eigenvalues of the Angular Momentum Operators

Today, we are talking about the eigenvalues of the angular momentum operators. **J** is used to denote angular momentum in general, **L** is used specifically to denote orbital angular momentum, and **S** is used to denote spin. **J= L+S** [Note: follows Section 9.2, page 358]

J is generalized; we still have the commuter relations:

$$\left.\begin{aligned}\left[\hat{J}_x,\hat{J}_y\right]&=i\hbar\hat{J}_z\\ \left[\hat{J}_y,\hat{J}_z\right]&=i\hbar\hat{J}_x\\ \left[\hat{J}_z,\hat{J}_x\right]&=i\hbar\hat{J}_y\end{aligned}\right\}\vec{J}=\vec{L}+\vec{S}=\text{total moment}$$

Today, we want to find the eigenvalues of J_z and J^2. The generalized relation is applicable to the z axis. We will show that:

$$\hat{J}_z=m\hbar \qquad \text{where} \quad m=-j,-j+1,...,0,j-1,j$$

$$\hat{J}^2=\hbar j(j+1) \qquad \text{where} \quad j=\tfrac{n}{2} \text{ and } n=0,1,2,...$$

Compare this to the orbital angular momentum operators:

$$\hat{L}_z=m\hbar \qquad \text{where} \quad m=-j,-j+1,...,0,j-1,j$$

$$\hat{L}^2=\hbar l(l+1) \qquad \text{where} \quad l=0,1,2,...$$

Ladder Operators

From the commutator relations and from the creation/annihilation operators from the Harmonic oscillator, we have the Ladder Operators for angular momentum.

$$\hat{J}_+=\hat{J}_x+i\hat{J}_y$$

$$\hat{J}_-=\hat{J}_x-i\hat{J}_y$$

$\hat{J}^2$ commutes with the otehr components of angular momentum

$$\left[\hat{J}_x,\hat{J}^2\right]=\left[\hat{J}_y,\hat{J}^2\right]=\left[\hat{J}_z,\hat{J}^2\right]=0$$

Some other properties of these operators:

$$\left.\begin{aligned}\left[\hat{J}_z,\hat{J}_+\right]&=\hbar\hat{J}_+\\ \left[\hat{J}_z,\hat{J}_-\right]&=\hbar\hat{J}_-\end{aligned}\right\}\text{We want to prove this directly.}$$

Here is our proof:
We show this explicitly:

$$\left[\hat{J}_z, \hat{J}_+\right] = \left[\hat{J}_z, \hat{J}_x + i\hat{J}_y\right]$$
$$= \hat{J}_z\left(\hat{J}_x + i\hat{J}_y\right) - \left(\hat{J}_x + i\hat{J}_y\right)\hat{J}_z$$
$$= \hat{J}_z\hat{J}_x + i\hat{J}_z\hat{J}_y - \hat{J}_x\hat{J}_z - i\hat{J}_x\hat{J}_y$$
$$= \left[\hat{J}_z, \hat{J}_x\right] + i\left[\hat{J}_z, \hat{J}_y\right]$$
$$= i\hbar\hat{J}_y - i^2\hbar\hat{J}_x$$
$$= \hbar\left(\hat{J}_x + i\hat{J}_y\right)$$
$$= \hbar\hat{J}_+$$

We can change the order to have a new relation:

$$\hat{J}_z\hat{J}_+ = \hat{J}_+\hat{J}_z + \hbar\hat{J}_+$$

Show that $\hat{J}^2 = \hat{J}_\mp\hat{J}_\pm + \hat{J}_z^2 \pm \hbar\hat{J}_z$

We want to prove this relation directly.

$$\hat{J}_-\hat{J}_+ = \left(\hat{J}_x - i\hat{J}_y\right)\left(\hat{J}_x + i\hat{J}_y\right)$$
$$= \hat{J}_x\hat{J}_x + \hat{J}_x i\hat{J}_y - i\hat{J}_y\hat{J}_x - i\hat{J}_y i\hat{J}_y$$
$$= \hat{J}_x{}^2 + i\hat{J}_x\hat{J}_y - i\hat{J}_y\hat{J}_x + \hat{J}_y{}^2$$
$$= \hat{J}_x{}^2 + \hat{J}_y{}^2 + i[\hat{J}_x, \hat{J}_y]$$
$$= \hat{J}_x{}^2 + \hat{J}_y{}^2 - \hbar\hat{J}_z$$
$$\therefore \hat{J}_x{}^2 + \hat{J}_y{}^2 + \hat{J}_z{}^2 = \hat{J}_-\hat{J}_+ + \hat{J}_z{}^2 + \hbar\hat{J}_z$$
$$\hat{J}^2 = \hat{J}_-\hat{J}_+ + \hat{J}_z{}^2 + \hbar\hat{J}_z$$

Prove the Commutator Relations

We can use these relations – put directly – to find the commutator relations.

$$\hat{J}^2 = \hat{J}_x{}^2 + \hat{J}_y{}^2 + \hat{J}_z{}^2$$
$$\hat{J}^2 = \hat{J}_\mp\hat{J}_\pm + \hat{J}_z^2 \pm \hbar\hat{J}_z$$

$$\left[\hat{J}_+, \hat{J}_-\right] = 2\hbar\hat{J}_z$$

$$\begin{aligned}\left[\hat{J}_+, \hat{J}_-\right] &= \hat{J}_+\hat{J}_- - \hat{J}_-\hat{J}_+ \\ &= \left(\hat{J}_x + i\hat{J}_y\right)\left(\hat{J}_x - i\hat{J}_y\right) - \left(\hat{J}_x - i\hat{J}_y\right)\left(\hat{J}_x + i\hat{J}_y\right) \\ &= \hat{J}_x\hat{J}_x - \hat{J}_x i\hat{J}_y + i\hat{J}_y\hat{J}_x - i\hat{J}_y i\hat{J}_y - \hat{J}_x\hat{J}_x - \hat{J}_x i\hat{J}_y + i\hat{J}_y\hat{J}_x + i\hat{J}_y i\hat{J}_y \\ &= 2i\left(\hat{J}_y\hat{J}_x - \hat{J}_x\hat{J}_y\right) \\ &= 2i\left[\hat{J}_y, \hat{J}_x\right]\end{aligned}$$

$$\therefore \left[\hat{J}_y, \hat{J}_x\right] = -i\hbar\hat{J}_z$$

$$\left[\hat{J}_x, \hat{J}_y\right] = i\hbar\hat{J}_z$$

Also we can show:

$$\hat{J}^2 = \hat{J}_-\hat{J}_+ + \hat{J}_z^2 + \hbar\hat{J}_z$$

$$+\hat{J}^2 = \hat{J}_+\hat{J}_- + \hat{J}_z^2 - \hbar\hat{J}_z$$

$$2\hat{J}^2 = \hat{J}_-\hat{J}_+ + \hat{J}_z^2 + \hbar\hat{J}_z + \hat{J}_+\hat{J}_- + \hat{J}_z^2 - \hbar\hat{J}_z$$

$$\therefore \boxed{\hat{J}_-\hat{J}_+ + \hat{J}_+\hat{J}_- = 2\left(\hat{J}^2 - \hat{J}_z^2\right)}$$

Eigenvalues of $\hat{J}^2$ and $\hat{J}_z$

To solve the problem of $\hat{J}^2$ and $\hat{J}_z$ to solve eigenvalue of these two physical parameters – first we see that they commute to each other.

$$\left[\hat{J}^2, \hat{J}_z\right] = 0$$

Here is the eigenvalue equation:

$$\hat{J}_z\varphi_m = \underbrace{\hbar m}_{\text{eigenvalue}}\ \overbrace{\varphi_m}^{\text{State function of } \hat{J}_z}$$

Use the commutator relations to change the order of these operators.

$$\left[\hat{J}_z, \hat{J}_+\right] = \hat{J}_z\hat{J}_+ - \hat{J}_+\hat{J}_z = \hbar\hat{J}_+ \Rightarrow \hat{J}_z\hat{J}_+ = \hat{J}_+\hat{J}_z + \hbar\hat{J}_+$$

$$\hat{J}_z\hat{J}_+\varphi_m = \left(\hat{J}_+\hat{J}_z + \hbar\hat{J}_+\right)\varphi_m = \hat{J}_+\hat{J}_z\varphi_m + \hbar\hat{J}_+\varphi_m = \hat{J}_+ m\hbar\varphi_m + \hbar\hat{J}_+\varphi_m = \hbar(m+1)\hat{J}_+\varphi_m$$

$$\hat{J}_z\left(\hat{J}_+\varphi_m\right) = \hbar(m+1)\left(\hat{J}_+\varphi_m\right)$$

So this implies that:

$$\hat{J}_+\varphi_m = \varphi_{m+1}$$

$$\hat{J}_z\hat{J}_+\varphi_m = \hbar(m+1)\varphi_{m+1}$$

Similarly:

$$\left[\hat{J}_z, \hat{J}_-\right] = \hat{J}_z\hat{J}_- - \hat{J}_-\hat{J}_z = -\hbar\hat{J}_- \Rightarrow \hat{J}_z\hat{J}_- = \hat{J}_-\hat{J}_z - \hbar\hat{J}_-$$

$$\hat{J}_z\hat{J}_-\varphi_m = \left(\hat{J}_-\hat{J}_z - \hbar\hat{J}_-\right)\varphi_m = \hat{J}_-\hat{J}_z\varphi_m - \hbar\hat{J}_-\varphi_m = \hat{J}_- m\hbar\varphi_m - \hbar\hat{J}_-\varphi_m = \hbar(m-1)\hat{J}_-\varphi_m$$

$$\hat{J}_z\left(\hat{J}_-\varphi_m\right) = \hbar(m-1)\left(\hat{J}_-\varphi_m\right)$$

So this implies that:

$$\hat{J}_-\varphi_m = \varphi_{m-1}$$

$$\hat{J}_z\hat{J}_-\varphi_m = \hbar(m-1)\varphi_{m-1}$$

Now we need to find the common eigenfunctions – the eigenfunction φ_m of $\hat{J}_z$ and the eigenfunction φ_k of $\hat{J}^2$. Common functions found from:

$$\hat{J}^2\varphi_m = \hbar^2 k^2\varphi_k \quad \text{where } k \text{ is any real integer}$$

We calculate the expectation value:

$$\left\langle J^2\right\rangle = \int \varphi_m \hat{J}^2\varphi_m dx = \hbar^2 k^2$$

also:

$$\left\langle J^2\right\rangle = \underbrace{\left\langle J_x^2\right\rangle}_{\geq 0} + \underbrace{\left\langle J_y^2\right\rangle}_{\geq 0} + \underbrace{\left\langle J_z^2\right\rangle}_{=\hbar^2 m^2}$$

$$\therefore \hbar^2 k^2 \geq \hbar^2 m^2$$

$$k^2 \geq m^2 \quad \text{if} \quad k > 0 \quad \text{then} \quad \underbrace{-k \leq m \leq k}_{\text{we have this confinement}}$$

m has a maximum and also a minimum

$$\hat{J}_+\varphi_{m\max} = 0 \quad \text{and} \quad \hat{J}_-\varphi_{m\min} = 0$$

$$\hat{J}^2\varphi_{m\max} = \hbar^2 k^2\varphi_{m\max} \quad \text{and} \quad \hat{J}^2\varphi_{m\min} = \hbar^2 k^2\varphi_{m\min}$$

To find the eigenvalues of $\hat{J}^2$ and $\hat{J}_z$ we use the commutator relations $\hat{J}^2 = \hat{J}_{\mp}\hat{J}_{\pm} + \hat{J}_z^2 \pm \hbar\hat{J}_z$:

$$\hat{J}^2\varphi_{m\max} = \left(\hat{J}_-\hat{J}_+ + \hat{J}_z^2 + \hbar\hat{J}_z\right)\varphi_{m\max} = \left(0 + \hbar^2 m_{\max}^2 + \hbar^2 m_{\max}\right)\varphi_{m\max} = \hbar^2 m_{\max}\left(m_{\max}+1\right)\varphi_{m\max}$$
$$\hat{J}^2\varphi_{m\max} = \hbar^2 k^2 \varphi_{m\max}$$
$$\hbar^2 k^2 \varphi_{m\max} = \hbar^2 m_{\max}\left(m_{\max}+1\right)$$
$$k^2 = \hbar^2 m_{\max}\left(m_{\max}+1\right)$$

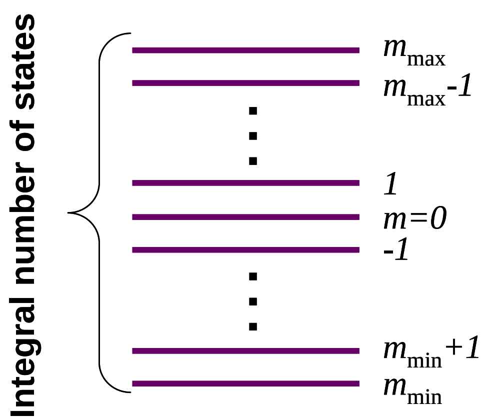
Look at this using the same method to do minimum.

$$\hat{J}^2\varphi_{m\min} = \left(\hat{J}_+\hat{J}_- + \hat{J}_z^2 - \hbar\hat{J}_z\right)\varphi_{m\min} = \left(0 + \hbar^2 m_{\min}^2 - \hbar^2 m_{\min}\right)\varphi_{m\min} = \hbar^2 m_{\min}\left(m_{\min}-1\right)\varphi_{m\min}$$
$$\hat{J}^2\varphi_{m\min} = \hbar^2 k^2 \varphi_{m\min}$$
$$\hbar^2 k^2 \varphi_{m\min} = \hbar^2 m_{\min}\left(m_{\min}-1\right)\varphi_{m\min}$$
$$k^2 = \hbar^2 m_{\min}\left(m_{\min}-1\right)$$

Let:

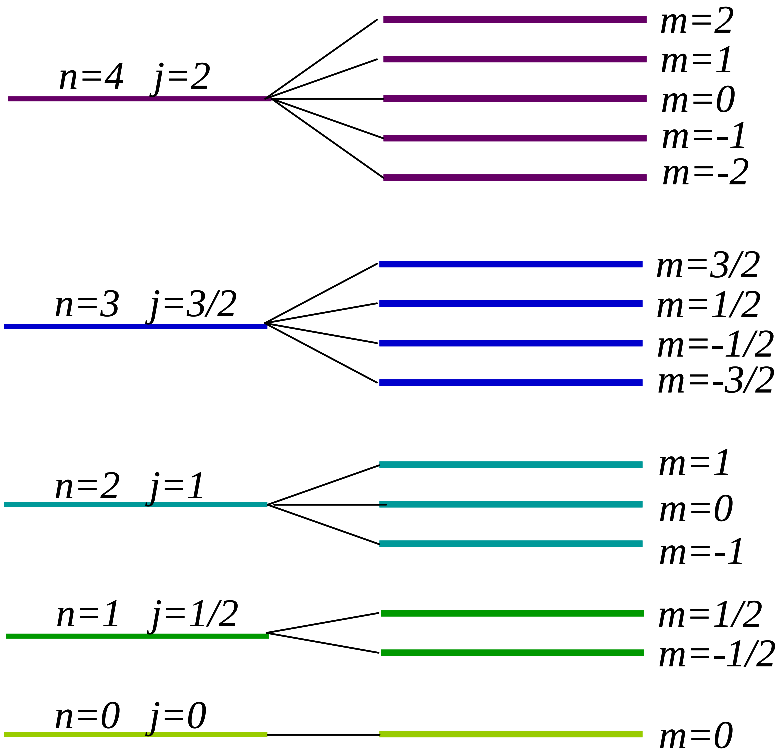
$$m_{\max} \equiv j$$

From the eigenfunction you have $J_z = \hbar m_j$. From the ladder operator, we have the range of values $m_j = \left[-j, -j+1, \ldots, j-1, j\right]$. Here are the values:



Here are the eigenvalues of $\hat{J}^2$ and $\hat{J}_z$. We got this result from the commutator relations $\hat{J}^2 = \hat{J}_{\mp}\hat{J}_{\pm} + \hat{J}_z^2 \pm \hbar\hat{J}_z$.

$J^2 = \hbar^2 j(j+1)$	$j = \frac{n}{2}$	$n = 0,1,2,\ldots$
$J_z = \hbar m$	$m = -j, -j+1, \ldots, 0, \ldots, j-1, j$	

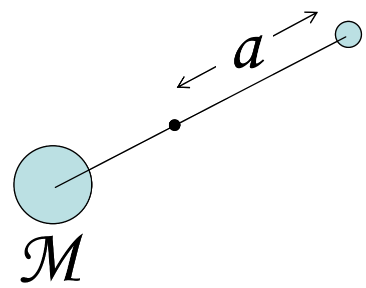


Similar to figure 9.6 on page 362.

Rotational Energy Levels of Molecules
The rotational level of a molecule is the lowest energy level – is rotation.

From classical mechanics:

$$\hat{H} = \frac{L^2}{2I} \quad \text{and} \quad I = 2a^2 M$$



We are not considering vibration, so a is constant. The eigenvalues are:

$$L^2 = \hbar^2 l(l+1)$$

$$E_l = \tfrac{1}{2I}\hbar^2 l(l+1) = \tfrac{1}{4Ma^2}\hbar^2 l(l+1)$$

$$E_{l+1} - E_l = \tfrac{1}{4Ma^2}\hbar^2\left[(l+1)(l+2) - l(l+1)\right] = \tfrac{1}{4Ma^2}\hbar^2\left[(l+1)(l+2-l)\right] = \tfrac{1}{2Ma^2}\hbar^2(l+1)$$

Using the same method, same eigenvalue equation – talk about Hydrogen – the simplest molecule M-M and the $\Delta E = E_l - E_{l-1}$. Here $\frac{\hbar^2}{2m_e a_{Bohr}^2} = 13.6\text{ eV}$ is called the Rydberg constant. It is the energy of ionizing a hydrogen atom.

If we consider this Rydberg constant to be $\frac{\hbar^2}{2Ma^2}$ is 13.6 eV$\left(\frac{m_e}{M}\right)$ where $\left(\frac{m_e}{M}\right)$ is usually on the order of 10^{-3}. So the Rydberg constant is about 10meV for Hydrogen (about 1K energy and $\lambda \to$mm. Remember for the qualifying exam – if you know this - $10 \cdot 10^{-3}$ eV.

Homework: 9.5 and 9.6 – not difficult – simple like general physics.