Today we are studying the annihilation and creation operators.

$$\psi_0 = \left(\frac{B}{\pi}\right)^2 e^{-B^2x^2/2}$$
$$\psi_1 = a^+\psi_0$$
$$\vdots$$
$$\psi_n = \frac{1}{\sqrt{n!}}\left(a^+\right)^n \psi_0$$

These are normalized wavefunctions; page 196 has the annihilation and creation operator relations:

$$1 = \langle\psi_1|\psi_1\rangle = \langle a^+\psi_0|a^+\psi_0\rangle = \langle\psi_0|aa^+\psi_0\rangle = \langle\psi_0|\left(1-a^+a\right)\psi_0\rangle = \langle\psi_0|\psi_0\rangle = 1$$

$$\langle\psi_2|\psi_2\rangle = \left\langle\left(a^+\right)^2\psi_0\middle|\left(a^+\right)^2\psi_0\right\rangle = \left\langle\psi_0\middle|(a)^2\left(a^+\right)^2\psi_0\right\rangle$$
$$aaa^+a^+ = a\left(1+a^+a\right)a^+$$
$$aa^+ + aa^+aa^+ = aa^+ + aa^+aa^+ = aa^+ + aa^+\left(aa^+ + 1\right) \quad \text{use this}$$

So, we can show that…

$$\psi_n = \left(a^+\right)^n \psi_0$$
$$\langle\psi_n|\psi_n\rangle = n! \text{ and } \langle\psi_{n+1}|\psi_{n+1}\rangle = (n+1)!$$
$$\psi_1 = a^+\psi_0 \quad \text{and} \quad \langle\psi_1|\psi_1\rangle = 1$$
$$\psi_2 = \left(a^+\right)^2\psi_0 \quad \text{and} \quad \langle\psi_2|\psi_2\rangle = 2$$

Using these relations:

$$\left\langle\psi_1\middle|(a)^n\left(a^+\right)^n\psi_n\right\rangle = n!$$
$$\left\langle\psi_0\middle|(a)^{n+1}\left(a^+\right)^{n+1}\psi_0\right\rangle = \left\langle\psi_0\middle|(a)^n aa^+\left(a^+\right)^n\psi_0\right\rangle = \left\langle\psi_0\middle|(a)^n\left(1+aa^+\right)\left(a^+\right)^n\psi_0\right\rangle$$
$$(a)^n\left(1+aa^+\right)\left(a^+\right)^n = (a)^n\left(a^+\right)^n + (a)^n aa^+a^+\left(a^+\right)^{n-1}$$
$$= \tfrac{n!+\cdots n!}{n+1} + (a)^n\left(a^+\right)^{n+1} a$$
$$= n!(n+1)$$
$$= (n+1)!$$

The creation/annihilation operators in Dirac notation:

$$a\left|\psi_n\right\rangle = (n)^{1/2}\left|\psi_{n-1}\right\rangle$$
$$a^+\left|\psi_n\right\rangle = (1+n)^{1/2}\left|\psi_{n+1}\right\rangle$$

← Memorize this!

Another way to write the Dirac notation:

$$a\left|n\right\rangle = (n)^{1/2}\left|n-1\right\rangle$$
$$a^+\left|n\right\rangle = (1+n)^{1/2}\left|n+1\right\rangle$$

Here is how to write the Hamiltonian with creation/annihilation operators:

$$\hat{H} = \hbar\omega_0\left(a^+a+\tfrac{1}{2}\right)$$
$$a^+a\left|n\right\rangle = a^+\left(a\left|n\right\rangle\right) = a^+\left(n^{1/2}\left|n-1\right\rangle\right) = n^{1/2}a^+\left|n-1\right\rangle = n^{1/2}n^{1/2}\left|n-1+1\right\rangle = n\left|n\right\rangle$$
$$\hat{H}\left|n\right\rangle = \hbar\omega_0\left(a^+a+\tfrac{1}{2}\right)\left|n\right\rangle = \left(n+\tfrac{1}{2}\right)\hbar\omega_0\left|n\right\rangle$$
$$\left\langle x\right\rangle = \tfrac{1}{2\sqrt{\beta}}\left(a+a^+\right) = \left\langle n\middle|xn\right\rangle = \left\langle n\middle|\left(a+a^+\right)n\right\rangle = \left\langle n\middle|an\right\rangle + \left\langle n\middle|a^+n\right\rangle$$
$$= \left\langle n\middle|n^{1/2}\middle|n-1\right\rangle + \left\langle n\middle|(n+1)^{1/2}\middle|n+1\right\rangle$$
$$= n^{1/2}\left\langle n\middle|n-1\right\rangle + (n+1)^{1/2}\left\langle n\middle|n+1\right\rangle = 0$$
$$\left\langle n\middle|m\right\rangle = \delta_{nm}$$
$$\left\langle p\right\rangle = 0$$
$$\left[a,a^+\right] = 1 \text{ for a fermion}$$

☼Homework: 7.8, 7.9, 7.10