Time development of State Functions

$t=0$ we have the time independent solution $\psi(x,0)$

$t>0 \quad \psi(x,t)=?$

$$\psi(x,t)=\exp\left(\frac{-i\hat{H}t}{\hbar}\right)\psi(x,t)$$

$$\exp\left(\frac{-i\hat{H}t}{\hbar}\right)=1-\frac{-i\hat{H}t}{\hbar}-\frac{\hat{H}^2t^2}{2!\hbar^2}+\cdots$$

If $\psi(x,t)$ is an eigenfunction ψ_n of $\hat{H}$

then $\exp\left(\frac{-i\hat{H}t}{\hbar}\right)\psi_n=\exp\left(\frac{-iE_n}{\hbar}\right)\psi_n$

t: stationary state

$$\psi_n(x,t)=\exp\left(\frac{-i\hat{H}t}{\hbar}\right)\psi_n(x)=\exp(-i\omega_n t)\psi_n(x)$$

$$\hbar\omega_n=E_n=n^2E_1$$

$\langle\psi_n|A\psi_n\rangle$ is a constant.

Example:

$\langle\psi_5|A\psi_5\rangle \quad n=5$

$$\psi_5(x,t)=\exp\left(\frac{-i25E_1t}{\hbar}\right)\sqrt{\frac{2}{a}}\sin\left(\frac{5\pi x}{a}\right)$$

$$\left\langle\psi_5(x,t)\middle|\hat{H}\middle|\psi_5(x,t)\right\rangle=$$

$$|\psi(x,0)\rangle=\sum_n b_n|\psi_n(x)\rangle$$

$$b_n=\langle\psi_n|\psi(x,0)\rangle$$

$$\psi(x,t)=\exp\left(\frac{-i\hat{H}t}{\hbar}\right)\sum_n b_n\psi_n=\sum_n b_n\exp\left(\frac{-i\hat{H}t}{\hbar}\right)\psi_n=\sum_n b_n\exp(-i\omega_n t)\psi_n=\sum_n\overline{b_n(t)}\psi_n$$

$$b_n(t)=\sum_n b_n\exp(-i\omega_n t)$$

$$\langle \psi_n(x,t) | \psi_n(x,t) \rangle = \sum_n |b_n|^2$$

$$\boxed{P(E_n) = \frac{|b_n|^2}{\sum_n |b_n|^2}} \leftarrow \text{Definition of probability of measurement}$$

Example:

$$\psi(x,0) = \sqrt{\frac{2}{a}}\left(\frac{\sin(2\pi x/a) + 2\sin(\pi x/a)}{\sqrt{5}}\right)$$

$$b_1 = \frac{2}{\sqrt{5}} \qquad b_2 = \frac{1}{\sqrt{5}} \qquad b_n = 0 \text{ for } n > 2$$

$$P(E_1) = |b_1|^2 = \frac{4}{5}$$

$$P(E_2) = |b_2|^2 = \frac{1}{5}$$

$$P(E_n) = |b_n|^2 = 0 \quad n > 2$$

$$\sum_n |b_n|^2 = \frac{4}{5} + \frac{1}{5} = 1$$

Expectation value of energy:

$$\langle E \rangle = \left\langle \psi(x,t) \middle| \hat{H}\psi(x,t) \right\rangle$$

$$= \left\langle \sum_n b_n \psi_n e^{-i\omega_n t} \middle| \sum_m b_m E_m \psi_m e^{-i\omega_m t} \right\rangle$$

$$= \sum_{nm} b_n^* b_m \langle \psi_n | \psi_m \rangle E_m$$

$$= \sum_{nm} b_n^* b_m \delta_{mn} E_m$$

$$= \sum_n |b_n|^2 E_n$$

Example in time:

$$\psi(x,t) = \sqrt{\frac{2}{a}}\left(\frac{\exp(-i\omega_2 t)\sin(2\pi x/a) + 2\exp(-i\omega_1 t)\sin(\pi x/a)}{\sqrt{5}}\right)$$

$$\tilde{b}_1(t) = \frac{2}{\sqrt{5}}\exp(-i\omega_1 t) \qquad \tilde{b}_2(t) = \frac{1}{\sqrt{5}}\exp(-i\omega_2 t) \qquad \bar{b}_n \equiv \exp(i\omega_n t) = 0 \text{ for } n > 2$$

Time Development of Expectation Values

$$-i\hbar\frac{\partial}{\partial t}\psi(x,t)=\hat{H}\psi(x,t)$$

$$\frac{d\left\langle\psi\middle|\hat{A}\psi\right\rangle}{dt}=\int dx\frac{\partial}{\partial t}\psi*\hat{A}\psi$$

$$\frac{\partial\psi*\hat{A}\psi}{\partial t}=\frac{\partial\psi*}{\partial t}\hat{A}\psi+\psi*\hat{A}\frac{\partial\psi}{\partial t}+\psi*\frac{\partial\hat{A}}{\partial t}\psi$$

$$\frac{\partial\psi}{\partial t}=\frac{-i\hat{H}}{\hbar}\psi$$

$$\frac{\partial\psi*}{\partial t}=\frac{-i\hat{H}}{\hbar}\psi*$$

$$\frac{\partial\left(\psi*\hat{A}\psi\right)}{\partial t}=\frac{i}{\hbar}$$

$$\left(\hat{H}\psi*\hat{A}\psi-\psi*\hat{A}\hat{H}\psi+\frac{\hbar}{i}\psi*\frac{\partial\hat{A}}{\partial t}\psi\right)$$

$$\frac{\partial\left\langle\hat{A}\right\rangle}{\partial t}=\frac{i}{\hbar}\left(\left\langle\hat{H}\psi\middle|\hat{A}\psi\right\rangle-\left\langle\psi\middle|\hat{A}\hat{H}\psi\right\rangle+-\frac{\hbar}{i}\left\langle\psi\middle|\frac{\partial\hat{A}}{\partial t}\hat{H}\psi\right\rangle\right)$$

$$\frac{\partial\left\langle\hat{A}\right\rangle}{\partial t}=\left\langle\frac{i}{\hbar}\left[\hat{H},\hat{A}\right]+\frac{\partial\hat{A}}{\partial t}\right\rangle$$

$$\frac{\partial\left\langle \hat{A}\right\rangle}{\partial t}=\frac{i}{\hbar}\left\langle\left[\hat{H},\hat{A}\right]\right\rangle$$

$$\hat{H}=\frac{p^2}{2m}+V(x)$$

$$\frac{\partial\left\langle x\right\rangle}{\partial t}=\frac{i}{\hbar}\left\langle\left[\hat{H},x\right]\right\rangle$$

$$=\frac{i}{\hbar}\left\langle\left[\frac{p^2}{2m},x\right]\right\rangle$$

$$=\frac{i}{2m}\left\langle p[p,x],+[p,x]p\right\rangle$$

$$=\frac{i}{2m\hbar}\left\langle -2i\hbar p\right\rangle$$

$$=\left\langle\frac{p}{m}\right\rangle$$

$$m\frac{\partial\left\langle x\right\rangle}{\partial t}=\left\langle p\right\rangle$$

$$\frac{\partial p}{\partial t}=\frac{i}{\hbar}\left\langle\left[\hat{H},p\right]\right\rangle=\frac{i}{\hbar}\left\langle\left[V(x),p\right]\right\rangle$$

$$V(x)\left(-i\hbar\frac{\partial}{\partial x}\right)g-\left(-i\hbar\frac{\partial}{\partial x}\right)V(x)g=\cancel{-i\hbar V(x)\frac{\partial g}{\partial x}+i\hbar V(x)\frac{\partial g}{\partial x}}+i\hbar g\frac{\partial V(x)}{\partial x}$$

$$\frac{\partial p}{\partial t}=i\hbar\frac{\partial V(x)}{\partial x}$$

$$\frac{\partial\left\langle p\right\rangle}{\partial t}=\frac{i}{\hbar}\left\langle\left[\hat{H},p\right]\right\rangle=\frac{i}{\hbar}\left\langle i\hbar\frac{\partial V(x)}{\partial x}\right\rangle=-\left\langle\frac{\partial V(x)}{\partial x}\right\rangle=\left\langle F\right\rangle$$

Ehrenfest's Principle: Quantum Mechanics $\xrightarrow[\text{average}]{}$ Classical Mechanics

☼ Homework: 6.1 (a) (1) and 6.10