Heisenberg's Uncertainty Principle

Heisenberg's uncertainty principle (sections 5.4, 5.5)

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

develop this formulation

– generalized using knowledge of linear algebra, etc.

$\hat{A}$ and $\hat{B}$ are Hermitian Operators

obey this relation: $[\hat{A},\hat{B}]=\hat{C}$

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}\left|\langle\hat{C}\rangle\right|$$

Today, we want to get this Cauchy-Schwartz Inequality

Consider 3-D vectors: $\vec{r}_1$ and $\vec{r}_2$

$$\left|\vec{r}_1 \cdot \vec{r}_2\right|^2 \leq \left|\vec{r}_1\right|^2 \left|\vec{r}_2\right|^2$$

Now consider State vectors: $|\varphi\rangle$ and $|\phi\rangle$

$$\left|\langle\varphi|\phi\rangle\right|^2 \leq \left|\langle\varphi|\varphi\rangle\right|^2 \left|\langle\phi|\phi\rangle\right|^2$$

Prove it! Dr. Shen likes his proof better than the book's proof.

λ is a non-zero complex number. It is the 'trick' for a simple proof.

This is Dr. Shen's Proof of the Cauchy-Swartz Inequality:

$$0 \le \underbrace{\langle \varphi - \lambda\phi | \varphi - \lambda\phi \rangle}_{\text{vector length}} = \int (\varphi - \lambda\phi)^* (\varphi - \lambda\phi)\, dx$$

$$= \int (\varphi^* - \lambda^*\phi^*)(\varphi - \lambda\phi)\, dx$$

$$= \int (\varphi^*\varphi - \lambda^*\phi^*\varphi - \varphi^*\lambda\phi + \lambda^*\phi^*\lambda\phi) dx$$

$$= \int \varphi^*\varphi dx - \lambda^* \int \phi^*\varphi dx - \lambda \int \varphi^*\phi dx + |\lambda|^2 \int \phi^*\phi dx$$

$$= \langle \varphi|\varphi \rangle - \lambda^* \langle \phi|\varphi \rangle - \lambda \langle \varphi|\phi \rangle + |\lambda|^2 \langle \phi|\phi \rangle$$

Here is the trick: $\lambda = \dfrac{\langle \phi|\varphi \rangle}{\langle \phi|\phi \rangle}$ and $\lambda^* = \dfrac{\langle \varphi|\phi \rangle}{\langle \phi|\phi \rangle}$

Substitute:

$$= \langle \varphi|\varphi \rangle - \frac{\langle \varphi|\phi \rangle \langle \phi|\varphi \rangle}{\langle \phi|\phi \rangle} - \frac{\langle \phi|\varphi \rangle \langle \varphi|\phi \rangle}{\langle \phi|\phi \rangle} + \frac{\langle \phi|\varphi \rangle \langle \varphi|\phi \rangle}{\langle \phi|\phi \rangle \cancel{\langle \phi|\phi \rangle}} \cancel{\langle \phi|\phi \rangle}$$

$$0 \le \langle \varphi|\varphi \rangle - \frac{|\langle \varphi|\phi \rangle|^2}{\langle \phi|\phi \rangle} - \frac{|\langle \varphi|\phi \rangle|^2}{\langle \phi|\phi \rangle} + \frac{|\langle \varphi|\phi \rangle|^2}{\langle \phi|\phi \rangle}$$

$$0 \le \langle \varphi|\varphi \rangle - \frac{|\langle \varphi|\phi \rangle|^2}{\langle \phi|\phi \rangle} \cancel{- \frac{|\langle \varphi|\phi \rangle|^2}{\langle \phi|\phi \rangle} + \frac{|\langle \varphi|\phi \rangle|^2}{\langle \phi|\phi \rangle}}$$

$$0 \le \langle \varphi|\varphi \rangle \langle \phi|\phi \rangle - |\langle \varphi|\phi \rangle|^2$$

$$|\langle \varphi|\varphi \rangle|^2 \le \langle \varphi|\varphi \rangle \langle \phi|\phi \rangle$$

Now prove the following:

$\hat{A}$ and $\hat{B}$ are Hermetian operators.

$\left[\hat{A},\hat{B}\right]=\hat{C}$

Prove that $\Delta A \Delta B \le \tfrac{1}{2}\left|\langle C\rangle\right|$

$$\left|\langle B\psi|A\psi\rangle\right|^2 \ge \langle A\psi|A\psi\rangle\langle B\psi|B\psi\rangle$$

$$\left|\langle B\psi|A\psi\rangle\right|^2 \ge \left|im\langle B\psi|A\psi\rangle\right|^2$$

$$\begin{aligned}\left|im\langle B\psi|A\psi\rangle\right|^2 &= \tfrac{1}{4}\left|2im\langle B\psi|A\psi\rangle\right|^2 \\ &= \tfrac{1}{4}\left|\langle B\psi|A\psi\rangle-\langle B\psi|A\psi\rangle *\right|^2 \\ &= \tfrac{1}{4}\left|\langle B\psi|A\psi\rangle-\langle A\psi|B\psi\rangle\right|^2 \\ &= \tfrac{1}{4}\left|\langle AB\psi|\psi\rangle-\langle BA\psi|\psi\rangle\right|^2 \\ &= \tfrac{1}{4}\left|\langle (AB-BA)\psi|\psi\rangle\right|^2 \\ &= \tfrac{1}{4}\left|\langle [AB,BA]\psi|\psi\rangle\right|^2 \\ &= \tfrac{1}{4}\left|\langle C\psi|\psi\rangle\right|^2\end{aligned}$$

$$\therefore \langle A\psi|A\psi\rangle\langle B\psi|B\psi\rangle \ge \tfrac{1}{4}\left|\langle C\psi|\psi\rangle\right|^2$$

Another way to prove that $\Delta A \Delta B \le \tfrac{1}{2}\left|\langle C\rangle\right|$

$$A' = (A-\langle A\rangle) = A-\Delta A$$

$$B' = (B-\langle B\rangle) = B-\Delta B$$

$$(\Delta A)^2 = \langle (A-\langle A\rangle)\psi|(A-\langle A\rangle)\psi\rangle$$

$$(\Delta B)^2 = \langle (B-\langle B\rangle)\psi|(B-\langle B\rangle)\psi\rangle$$

apply into $\langle A\psi|A\psi\rangle\langle B\psi|B\psi\rangle \ge \tfrac{1}{4}\left|\langle C\psi|C\psi\rangle\right|^2$

$$\left|\langle C\rangle\right| = \left|\langle C\psi|\psi\rangle\right|$$

$$(\Delta A)^2(\Delta B)^2 \ge \tfrac{1}{4}\left|(\Delta C)\right|^2$$

Understand this and be able to do it at home. Page 148 is not as good.

Heisenberg's Uncertainty Relation:

$$p_x = -i\hbar \frac{\partial}{\partial x} \Rightarrow [x, p_x] = i\hbar$$

$$|\langle i\hbar \rangle|^2 = \hbar^2$$

$$(\Delta x)^2 (\Delta p_x)^2 \geq \frac{\hbar^2}{4}$$

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}$$

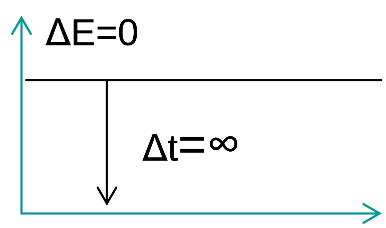
Also Heisenberg's Uncertainty Relation in terms of Energy Operator:

$$E = i\hbar \frac{\partial}{\partial t}$$

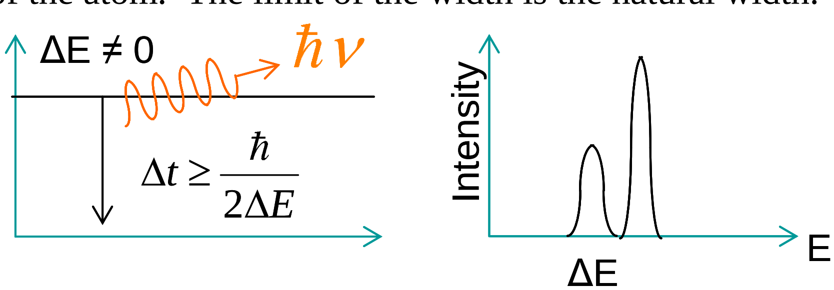
$$[t, E] = ?$$

$$\Delta t \Delta E \geq \frac{\hbar}{2}$$

If you have a quantum state for which ΔE=0, then e- remains on this state forever:



Consider the excited state of the sodium atom. There is a spectral width of the line. The excited state emits light when it returns to ground state. The reason for the broadness of the line is the Doppler effect which is an intrinsic property of the atom. The limit of the width is the natural width.



☼Homework: On Handout - Prove Cauchy-Schwartz inequality, Prove Roberston-Schrodinger equation, 5.28