Commutator Relations

$\hat{\mathbf{A}}, \hat{\mathbf{B}} = \text{operator}$

$\left[\hat{\mathbf{A}}, \hat{\mathbf{B}}\right] = \hat{\mathbf{A}}\hat{\mathbf{B}} - \hat{\mathbf{B}}\hat{\mathbf{A}}$

If $\left[\hat{\mathbf{A}}, \hat{\mathbf{B}}\right] = 0$

then $\hat{\mathbf{A}}\hat{\mathbf{B}} = \hat{\mathbf{B}}\hat{\mathbf{A}}$ and $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ commute with each other

Important Commutator Relations

Any operator $\hat{\mathbf{A}}$ commutes wtih any constant a

$\left[\hat{\mathbf{A}}, a\right] = 0$

$\left[\hat{\mathbf{A}}, a\right] = \hat{\mathbf{A}}a - a\hat{\mathbf{A}} = a\hat{\mathbf{A}} - \hat{\mathbf{A}}a = 0$

$\left[\hat{\mathbf{A}}, a\hat{\mathbf{B}}\right] = \left[a\hat{\mathbf{A}}, \hat{\mathbf{B}}\right] = a\left[\hat{\mathbf{A}}, \hat{\mathbf{B}}\right]$

Any $\hat{\mathbf{A}}$ operator commutes with its own square $\hat{\mathbf{A}}^2$

$\left[\hat{\mathbf{A}}, \hat{\mathbf{A}}^2\right] = \hat{\mathbf{A}}\hat{\mathbf{A}}^2 - \hat{\mathbf{A}}^2\hat{\mathbf{A}} = \hat{\mathbf{A}}^3 - \hat{\mathbf{A}}^3 = 0$

Another way to generalize the commutation of $\hat{\mathbf{A}}$ with $\hat{\mathbf{A}}^n$

$\left[\hat{\mathbf{A}}, \hat{\mathbf{B}}\right]g(x) = \hat{\mathbf{A}}\hat{\mathbf{B}}g(x) - \hat{\mathbf{B}}\hat{\mathbf{A}}g(x)$

$\left[\hat{\mathbf{A}}, \hat{\mathbf{A}}^2\right]g(x) = 0 \Leftarrow \text{important relation}$

Any operator $\hat{\mathbf{A}}$ commutes with any function of $\hat{\mathbf{A}}$, $f(\hat{\mathbf{A}})$

$\left[f\left(\hat{\mathbf{A}}\right), \hat{\mathbf{A}}\right] = \left[\hat{\mathbf{A}}, f\left(\hat{\mathbf{A}}\right)\right]0$

An example of commutation with function:

$[e^{\hat{p}}, \hat{p}] = \left[\sum_{n=0}^{\infty} \frac{\hat{p}^n}{n!}, \hat{p}\right] = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\hat{p}^n, \hat{p}\right] = [1, \hat{p}] + [\hat{p}, \hat{p}] + \frac{1}{2!}[\hat{p}^2, \hat{p}] + \cdots = 0$

Chain rule of differentiation:

$[f(x), g(x)]' = [f'(x), g(x)] + [f(x), g'(x)]$

Do $\hat{x}$ and $\hat{p}$ commute?

What is their commutator relation?

$$[\hat{x}, \hat{p}] = ?$$

$$[\hat{x}, \hat{p}]g(x) = \hat{x}\hat{p}g(x) - \hat{p}\hat{x}g(x)$$

$$= x\left(-i\hbar\frac{\partial}{\partial x}\right)g(x) - \left(-i\hbar\frac{\partial}{\partial x}\right)xg(x)$$

$$= -i\hbar x\frac{\partial g(x)}{\partial x} + i\hbar\frac{\partial xg(x)}{\partial x}$$

$$= -i\hbar xg' + i\hbar(xg' + g)$$

$$= i\hbar g(x)$$

$$\therefore [\hat{x}, \hat{p}] = i\hbar$$

Do $\hat{x}$ and $\hat{p}^2$ commute?

What is their commutator relation?

$$[\hat{x}, \hat{p}^2] = ?$$

$$[\hat{x}, \hat{p}^2] = \hat{x}\hat{p}\hat{p} - \hat{p}\hat{p}\hat{x}$$

$$= \hat{x}\hat{p}\hat{p} - \hat{p}\hat{x}\hat{p} + \hat{p}\hat{x}\hat{p} - \hat{p}\hat{p}\hat{x}$$

$$= (\hat{x}\hat{p} - \hat{p}\hat{x})\hat{p} + \hat{p}(\hat{x}\hat{p} - \hat{p}\hat{x})$$

$$= [\hat{x}, \hat{p}]\hat{p} + \hat{p}[\hat{x}, \hat{p}]$$

$$= i\hbar\hat{p} + \hat{p}i\hbar$$

$$= 2i\hbar\hat{p}$$

$$\therefore [\hat{x}, \hat{p}^2] = 2i\hbar\hat{p}$$

Do $\hat{x}^2$ and $\hat{p}$ commute?

$$[\hat{x}^2, \hat{p}] = \hat{x}\hat{x}\hat{p} - \hat{p}\hat{x}\hat{x}$$

$$= \hat{x}\hat{x}\hat{p} - \hat{x}\hat{p}\hat{x} + \hat{x}\hat{p}\hat{x} - \hat{p}\hat{x}\hat{x}$$

$$= \hat{x}[\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]\hat{x}$$

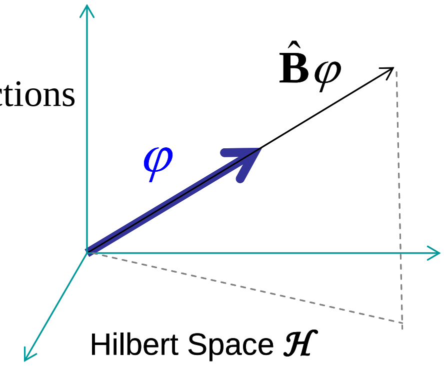
$$= \hat{x}i\hbar + \hat{x}i\hbar$$

$$= 2i\hbar\hat{x}$$

$$\therefore [\hat{x}^2, \hat{p}] = 2i\hbar\hat{x}$$

If $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ commute with eachother $\hat{\mathbf{A}}\hat{\mathbf{B}} = \hat{\mathbf{B}}\hat{\mathbf{A}}$

Then $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ have a common set of eigenfunctions



Prove this:

$\hat{\mathbf{A}}\varphi = a\varphi$

$\hat{\mathbf{B}}(\hat{\mathbf{A}}\varphi) = a\hat{\mathbf{B}}\varphi$

$\hat{\mathbf{A}}(\hat{\mathbf{B}}\varphi) = a\hat{\mathbf{B}}\varphi$

$\therefore \hat{\mathbf{B}}\varphi$ is an eigenfunction of $\hat{\mathbf{A}}$ corresponding to eigenvalue a

If φ is the only linear independent eigenfunction of $\hat{\mathbf{A}}$ that corresponds to eigenvalue a, then $\hat{\mathbf{B}}\varphi$ differs by only a multiplicative constant:

$\hat{\mathbf{B}}\varphi = \mu\varphi$

$\therefore$ φ is also an eigenfunction of $\hat{\mathbf{B}}$

Linear Independent - if $\varphi(x) = \sum_n c_n \varphi_n(x) = 0$ for all x, then $c_n = 0$ for all n.

Check this theorem:

The free particle momentum $\hat{p}$ term and Hamiltonian $\hat{H}$ have a common set of eigenfunctions.

Prove this:

$\hat{H} = \frac{\hat{p}^2}{2m}$ and $[\hat{p},\hat{H}]=0$

$\hat{p}\frac{\hat{p}^2}{2m} - \frac{\hat{p}^2}{2m}\hat{p} = 0$

$\varphi_k = Ae^{ikx}$

$$\left.\begin{aligned} \hat{p}\varphi_k &= i\hbar\varphi_k \\ \hat{H}\varphi_k &= \frac{\hat{p}^2}{2m}\varphi_k = \frac{\hbar^2 k^2}{2m}\varphi_k \end{aligned}\right\} \text{have a common set of eigenfunctions}$$

The definition of the Parity operator: $\hat{\wp}\,\varphi(x) = \varphi(-x)$

Suppose that $\varphi(x)$ is an eigenfunction.

$\hat{\wp}\,\varphi(x) = \varphi(-x)$

$\hat{\wp}\,\varphi(x) = \alpha\varphi(x) = \varphi(-x)$

$\hat{\wp}^2\varphi(x) = \alpha^2\varphi(x) = \varphi(x)$

$\alpha^2 = 1$

$\alpha = \pm 1$

$\hat{\wp}\,\varphi_{even}(x) = \varphi_{even}(-x) = \varphi_{even}(x)$

$\hat{\wp}\,\varphi_{odd}(x) = \varphi_{odd}(-x) = -\varphi_{odd}(x)$

Consider symmetric system V=∞ $|x| \geq a/2$ and V=0 $-a/2 < x < a/2$

$\hat{H} = \dfrac{\hat{p}^2}{2m} + V(x)$

$\hat{\wp}\,V(x) = V(-x) = V(x)$

If you find that

$\hat{\wp}\,V(x)\varphi(x) = V(-x)\varphi(-x) = V(x)\hat{\wp}\,\varphi(x)$

$[\hat{\wp}, V(x)] = 0$

Prove this:

$\left[\hat{\wp}, \dfrac{p^2}{2m}\right] = 0$

$[\hat{\wp}, p^2] = 0$

$\hat{\wp}\,\hat{p}\varphi(x) = \hat{\wp}\left(-i\hbar\nabla\right)\varphi(x) = -i\hbar\dfrac{\partial}{\partial(-x)}\hat{\wp}\,\varphi(x) = i\hbar\dfrac{\partial}{\partial x}\hat{\wp}\,\varphi(x) = -\hat{p}\hat{\wp}\,\varphi(x)$

$\therefore [\hat{\wp}, \hat{p}] = -1$

$\hat{\wp}\,\hat{p} + \hat{p}\hat{\wp} = 0$

$\hat{\wp}\,\hat{p}\hat{p} - \hat{p}\hat{p}\hat{\wp} = -\hat{p}\hat{\wp}\,\hat{p} - \hat{p}\hat{p}\hat{\wp} + \hat{p}\hat{p}\hat{\wp} - \hat{p}\hat{p}\hat{\wp} = 0$

Hamiltonian with a Symmetric Potential
Commutes with the Parity Operator

Hamiltonian from the Schrödinger Equation:

$$\frac{\partial^2}{\partial x^2}\varphi(x) = k^2\varphi(x)$$

$$\frac{\hbar^2 k^2}{2m} = E$$

eigenfunctions should be even or odd

$$\varphi = A\cos(kx)$$

$$parity = -1$$

$$\frac{ka}{2} = \frac{n\pi}{2} \qquad n = 1,3,5,\cdots$$

$$E_n = \frac{\hbar^2\left(\frac{n\pi}{a}\right)^2}{2m}$$

$$\varphi = A\sin(kx)$$

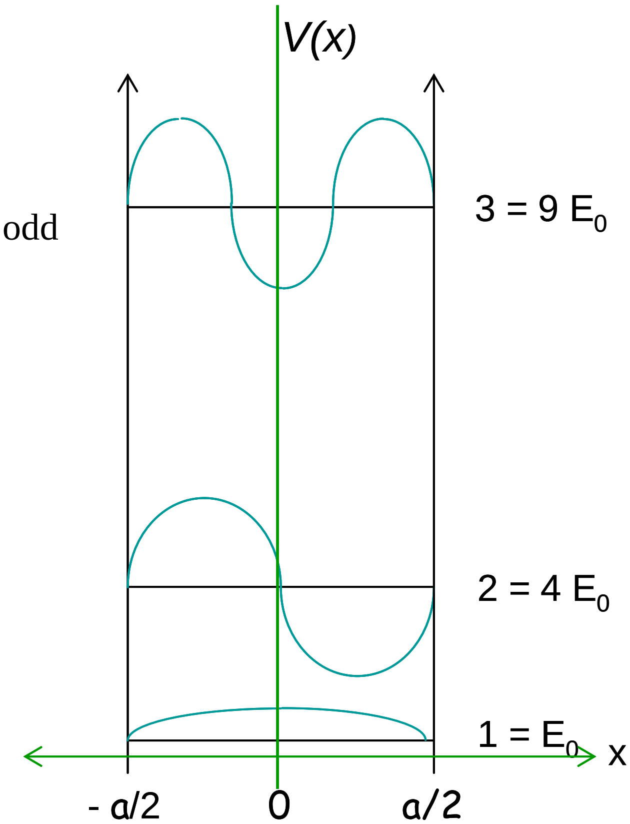
$$parity = -1$$

$$\frac{ka}{2} = n\pi$$

$$k = \frac{2n\pi}{a} \qquad n = 2,4,6,\cdots$$

$$E_n = \frac{\hbar^2\left(\frac{n\pi}{a}\right)^2}{2m}$$

also useful for finite well - very important
to treat or simplify problems.



☼ **Homework: 5.12, 5.13, 6.16 and**

Show that if $\hat{C}\psi = \psi^*$ $\hat{C}$ is not Hermitian

This proof shows that the contrary assumption leads to contradiction. Start with definition of Hermitian operator. An operator is Hermitian if and only if:

$$\int_{-\infty}^{\infty} dx \phi_n^*(x) \hat{\mathbf{O}} \psi_m(x) = \left[\int_{-\infty}^{\infty} dx \psi_m^*(x) \hat{\mathbf{O}} \phi_n(x) \right]^*$$

$$\int_{-\infty}^{\infty} dx \phi_n^*(x) \hat{\mathbf{C}} \psi_m(x) = \int_{-\infty}^{\infty} dx \phi_n^*(x) \psi_m^*(x)$$

$$\left[\int_{-\infty}^{\infty} dx \psi_m^*(x) \hat{\mathbf{C}} \phi_n(x) \right]^* = \left[\int_{-\infty}^{\infty} dx \psi_m^*(x) \phi_n^*(x) \right]^* = \int_{-\infty}^{\infty} dx \phi_n(x) \psi_m(x)$$

since

$$\int_{-\infty}^{\infty} dx \phi_n^*(x) \psi_m^*(x) \neq \int_{-\infty}^{\infty} dx \phi_n(x) \psi_m(x)$$

$$\int_{-\infty}^{\infty} dx \phi_n^*(x) \hat{\mathbf{C}} \psi_m(x) \neq \left[\int_{-\infty}^{\infty} dx \psi_m^*(x) \hat{\mathbf{C}} \phi_n(x) \right]^*$$

$\therefore \hat{\mathbf{C}}$ is not Hermitian

What are the eigenfunctions of $\hat{\mathbf{C}}$? (c) What are the eigenvalues of $\hat{\mathbf{C}}$?

$$\hat{\mathbf{C}} \varphi(x) = \varphi^*(x)$$

$$\hat{\mathbf{C}} \varphi(x) = c \varphi(x)$$

$$\therefore c \varphi(x) = \varphi^*(x)$$

$$\hat{\mathbf{C}}^2 \varphi(x) = \hat{\mathbf{C}} \hat{\mathbf{C}} \varphi(x) = \hat{\mathbf{C}} \varphi^*(x) = \varphi(x)$$

$$\hat{\mathbf{C}}^2 \varphi(x) = c^2 \varphi(x)$$

$$\therefore c^2 = 1$$

$$c = \pm 1$$

since $c\varphi(x) = \varphi^*(x)$

if $\varphi(x) = \mathrm{Re}(\Psi)$, then $\mathrm{c} = 1$

if $\varphi(x) = \mathrm{Im}(\Psi)$, then $\mathrm{c} = i$ where $\Psi =$ complex function in Hilbert Space.