Homework Set # 1   95.338/538

Due next week/class

<u>Please write legibly and show the steps</u>

Given Vectors : $\mathbf{A} = \mathbf{i}\, x^2 y + \mathbf{j} y^2 z + \mathbf{k} x z^2$

$\mathbf{B} = \mathbf{i}\, 2 + \mathbf{j} 2 + \mathbf{k} 4$ $\mathbf{C} = \mathbf{B} = \mathbf{i}\, 3 + \mathbf{j} 2 + \mathbf{k}$ $f = x^2 y^2 z$

1. Calculate **BxC** ; **CxB**; Angle between vectors **C** and **B** ; Divergence of $\vec{A}$ ($\nabla \cdot \bar{A}$) ; Curl of **A** ($\bar{\nabla} \times \bar{A}$); Grad f(∇f) ; Laplacian f ($\nabla^2 f$).

$\vec{C} \cdot \vec{B} = CB\cos\theta$

2. Prove the following: a) Curl of a gradient ($\nabla \times \nabla \phi$) = 0

b) Divergence of a curl of a vector ($\nabla \cdot (\nabla \times \bar{M})$) = 0

c) $\bar{\nabla} \times \bar{\nabla} \times \bar{A} = \bar{\nabla}(\bar{\nabla} \cdot \bar{A}) - \nabla^2 \bar{A}$

ϕ, $\bar{M}$ and $\bar{A}$ are arbitrary ~~fun~~ scalar function and vectors.

$\frac{20}{20}$

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$\vec{A} = x^2y\hat{i} + y^2z\hat{j} + xz^2\hat{k}$ $\vec{B} = 2\hat{i} + 2\hat{j} + 4\hat{k}$ $\vec{C} = 3\hat{i} + 2\hat{j} + \hat{k}$ $f = x^2y^2z$

1.) $\vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 4 \\ 3 & 2 & 1 \end{vmatrix} = \hat{i}(2-8) - \hat{j}(2-12) + \hat{k}(4-6) = -6\hat{i} + 10\hat{j} - 2\hat{k}$

$\vec{C}\times\vec{B} = -\vec{B}\times\vec{C} = 6\hat{i} - 10\hat{j} + 2\hat{k}$

$\psi = \cos^{-1}\frac{\vec{C}\cdot\vec{B}}{|\vec{C}||\vec{B}|} = \cos^{-1}\left[\frac{6+4+4}{\sqrt{4+4+16}\sqrt{9+4+1}}\right] = \cos^{-1}\left[\frac{14}{\sqrt{24}\sqrt{14}}\right] = 0.702\text{rad} = 40.2°$

$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(y^2z) + \frac{\partial}{\partial z}(xz^2) = 2xy + 2yz + 2xz = 2(xy + yz + xz)$

$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & y^2z & xz^2 \end{vmatrix} = \hat{i}(0-y^2) + \hat{j}(z^2-0) + \hat{k}(0-x^2) = -y^2\hat{i} - z^2\hat{j} - x^2\hat{k}$

$\vec{\nabla}f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = \hat{i}\,2xy^2z + \hat{j}\,2x^2yz + \hat{k}\,x^2y^2$

$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 2y^2z + 2x^2z + 0 = 2z(x^2+y^2)$

$\frac{10}{10}$

2) Prove:

a) Curl of Gradient is zero:

$\vec{\nabla}\times\vec{\nabla}\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix} = \left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right)\hat{i} + \left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right)\hat{j} + \left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)\hat{k} = 0$

(assuming continuous partial derivatives)

b) Divergence of a curl of a vector is zero:

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{m}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M_x & M_y & M_z \end{vmatrix} = \cancel{\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)}$

$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left[\left(\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right)\hat{i} + \left(\frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}\right)\hat{j} + \left(\frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}\right)\hat{k}\right]$

$= \left(\frac{\partial^2 M_z}{\partial x\partial y} - \frac{\partial^2 M_y}{\partial x\partial z}\right) + \left(\frac{\partial^2 M_x}{\partial y\partial z} - \frac{\partial^2 M_z}{\partial y\partial x}\right) + \left(\frac{\partial^2 M_y}{\partial z\partial x} - \frac{\partial^2 M_x}{\partial z\partial y}\right)$

$= \frac{\partial^2 M_z}{\partial x\partial y} - \frac{\partial^2 M_z}{\partial y\partial x} - \frac{\partial^2 M_y}{\partial x\partial z} + \frac{\partial^2 M_y}{\partial z\partial x} + \frac{\partial^2 M_x}{\partial y\partial z} - \frac{\partial^2 M_x}{\partial z\partial y} = 0$

assuming continuous partial derivatives

$\rightarrow$

c) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}\times\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ A_x & A_y & A_z\end{vmatrix} = \vec{\nabla}\times\left\{\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\hat{\imath}+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\hat{\jmath}+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\hat{k}\right\}$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right) & \left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right) & \left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\end{vmatrix}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \hat{\imath}\left[\frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\right]$$
$$+\hat{\jmath}\left[\frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)-\frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\right]$$
$$+\hat{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\right]$$
$$=\hat{\imath}\left[\left(-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial x^2}\right)A_x+\frac{\partial^2 A_y}{\partial x\partial y}+\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_z}{\partial z\partial x}\right]$$
$$+\hat{\jmath}\left[\left(-\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)A_y+\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_z}{\partial z\partial y}+\frac{\partial^2 A_x}{\partial x\partial y}\right]$$
$$+\hat{k}\left[\left(-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}\right)A_z+\frac{\partial^2 A_x}{\partial x\partial z}+\frac{\partial^2 A_y}{\partial y\partial z}+\frac{\partial^2}{\partial z^2}A_z\right]$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\nabla^2 A_x\hat{\imath}-\nabla^2 A_y\hat{\jmath}-\nabla^2 A_z\hat{k}$$
$$+\hat{\imath}\frac{\partial}{\partial x}\left[\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right]+\hat{\jmath}\frac{\partial}{\partial y}\left[\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)+\hat{k}\frac{\partial}{\partial z}\left[\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right]$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = -\nabla^2\left(A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}\right]+\hat{\imath}\frac{\partial}{\partial x}(\vec{\nabla}\cdot\vec{A})+\hat{\jmath}\frac{\partial}{\partial y}(\vec{\nabla}\cdot\vec{A})+\hat{k}\frac{\partial}{\partial z}(\vec{\nabla}\cdot\vec{A})$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = -\nabla^2\vec{A}+\left[\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right]\vec{\nabla}\cdot\vec{A}$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A}$$