3.1 Understanding Hermite-Gaussian beam modes.

a) What is the physical significance of the distance z_0?

z_0 is the characteristic length parameter of a Gaussian beam. z_0 = scale length

$$\Psi_0(z=0) = e^{-kr^2/2z_0} e^{-jP(z=0)}$$

$$z_0 = \frac{\pi n w_0^2}{\lambda_0}$$

The exponential term is real, amplitude drops off quite rapidly with r, being down from its peak value of 1 at $r=0$ to 0.368 at $r = (2z_0/k)^{1/2}$.

b) If $z = z_0$ and $r^2 = w^2(z_0)$, by how much does the phase of the field lead or lag that at $r=0$?

the radial phase factor is

$$\phi_r = e^{-jkr^2/2R(z)}$$

$$\text{here} \quad r^2 = w^2(z_0) = w_0^2\left[1+\left(\frac{z}{z_0}\right)^2\right]\Big|_{z_0} = 2w_0^2 = \frac{2z_0}{k}\left[1+\left(\frac{z}{z_0}\right)^2\right] = \frac{4z_0}{k}$$

$$R(z)\Big|_{z_0} = z\left(1+\frac{z^2}{z^2}\right)^{1/2}\Big|_{z_0} = z_0\left(1+\frac{z_0^2}{z_0^2}\right)^{1/2} = \sqrt{2}\,z_0$$

$$\frac{r^2}{R(z)} = \cancel{\frac{2w_0^2}{z_0\left(1+2w_0/z_0\right)^{1/2}}} \quad \frac{4z_0}{k\sqrt{2}z_0} = \frac{4\sqrt{2}}{2k} = \frac{2\sqrt{2}}{k}$$

$$\text{also} \quad w_0^2 = \frac{2z_0}{k} = \frac{\lambda_0 z_0}{n\pi} \qquad z_0 = \frac{\pi n w_0^2}{\lambda_0} \qquad |k| = \frac{2\pi}{\lambda_0}$$

$$\phi_r = \frac{kr^2}{2R(z)} = \cancel{\frac{4z_0 k}{2k\, z_0\left(1+4z_0/kz_0^2\right)^{1/2}}} \quad \frac{2\sqrt{2}}{k}\,\frac{k}{2} = \sqrt{2}$$

~~$\phi_r = e^{j\sqrt{2}}$~~ ~~thus~~ the phase factor $= e^{-j\sqrt{2}} = e^{-j\phi_r}$

c) Which factor expresses the idea that the beams are not plane waves and the phase velocity is greater than c?

the longitudinal phase factor $e^{-j\phi_e}$

$$\phi_e = kz - \tan^{-1}\left(\frac{z}{z_0}\right)$$

.2 (a) HeNe laser has far-field divergence 1 mrad at $\lambda_0 = 632.8\,nm$. What is the spot size w_0?

$$\theta/2 = 1\,mrad \qquad \lambda_0 = 632.8\,nm$$

$$\theta/2 = w_0/z_0 = \frac{w_0}{\pi w_0^2/\lambda_0} = \frac{\lambda_0}{\pi w_0}$$

$$w_0 = \frac{\lambda_0}{\pi(\theta/2)} = \frac{632.8\,nm}{\pi\, 10^{-3}\,rad} = 201{,}426\,nm = 201.4\,\mu m$$

) The power emitted by this laser is 5mW. What is the peak electric field in V/cm at $r=z=0$?

$$P = 5\,mW$$

$$I_0 = \frac{P_0}{\pi w_0^2/2} = \frac{5\cdot 10^{-3}\,W}{\pi(201.4\cdot 10^{-6}\,m)^2/2} = 78{,}475\,\frac{W}{m^2}$$

$$I_0 = 78{,}475\left(\frac{W}{m^2}\right)\left(\frac{1m}{100cm}\right)^2 = 7.85\,W/cm^2$$

$$E_0 = \sqrt{\frac{2Z_0 I_0}{n}} = \sqrt{\frac{2\cdot 377\Omega\cdot 7.85\,W/cm^2}{1}}$$

$$E_0 = 76.9\,V/cm$$

(c) How many photons per second are emitted by this laser beam?

$$\mathcal{N} = \frac{P}{h\nu} = \frac{P\lambda_0}{hc} = \frac{(5\cdot 10^{-3}\,W)(632.8\cdot 10^{-9}\,m)}{(6.626\cdot 10^{-34}\,Js)(3\cdot 10^8\,m/s)}.$$

$$\mathcal{N} = 1.59\cdot 10^{16}\ \text{photons/s}$$

(d) $P = \mathcal{N} h\nu$ If one more photon/s, how much more power?

$$\frac{dP}{d\mathcal{N}} = h\nu = \frac{hc}{\lambda_0} = \frac{(6.626\cdot 10^{-34}\,Js)(3\cdot 10^8\,m/s)}{(632.8\cdot 10^{-9}\,m)}$$

$$\Delta P \rightarrow 3.14\cdot 10^{-19}\,W$$

3.3 Given a 1-W $TEM_{0,0}$ beam of $\lambda_0 = 514.5\,nm$ from an argon ion laser with a minimum spot size of $w_0 = 2mm$ located at $z=0$.

(a) How far will this beam propagate before the spot size is 1cm?

$$W^2(z) = \frac{2}{kz_0}(z_0^2 + z^2) \Rightarrow z^2 = \frac{1}{2} kz_0 W^2 - z_0^2$$

$k = \omega/u = \omega n/c = 2\pi n/\lambda_0 \qquad z_0 = \pi n w_0^2/\lambda_0$

$\cancel{z^2 = \frac{1}{2}\left(\frac{2\pi n}{\lambda_0 z_0}\right) W^2 - z_0^2}$ $\qquad z_0 = \pi(2\cdot10^{-3})^2/(514.5\cdot10^{-9})\ m$

$z_0 = 24.42\,m$

$\cancel{z = \left(\frac{\pi n}{z_0\lambda} W^2 - z_0^2\right)^{1/2} = \left(\frac{\pi n}{z_0\lambda_0}\left[W^2 - w_0^2\right]\right)^{1/2}}$

$\cancel{z = \frac{\pi(1.0)}{514.5\cdot10}\left[(1\cdot10^{-2})^2\right.} \quad \cancel{z = \left[\left(\frac{\pi n}{\lambda_0}\right)\left(\frac{\lambda_0}{\pi n w_0^2}\right)\left[W^2 - w_0^2\right]\right]^{1/2}}$

$$kz_0 = \left(\frac{2\pi n}{\lambda_0}\right)\left(\frac{\pi n w_0^2}{\lambda_0}\right) = 2n^2\pi^2 w_0^2/\lambda_0^2$$

$$z^2 = \frac{1}{2} kz_0(W^2 - z_0^2) = \cancel{\frac{n^2\pi^2 w_0^2}{\lambda_0}\left(W^2 - \frac{\pi^2 n^2 w_0^4}{\lambda_0^2}\right)}$$

$$kz_0 = 2n^2\pi^2 w_0^2/\lambda_0^2 = 2\pi^2(2\cdot10^{-3}m)^2/(514.5\cdot10^{-9}m)^2 = 2.983\cdot10^8$$

$\cancel{z = \frac{1}{2}(2.983\cdot10^8)(1\cdot10^{-2}m)^2 - (24.42m)^2)^{1/2}}$

$$z^2 = \frac{1}{2}(2.983\cdot10^8)(1\cdot10^{-2}m)^2 - (24.42\,m)^2$$

$$z^2 = 14{,}915\,m^2 - 596.3\,m^2 = 14318.7\,m^2$$

$$z = 119.7\,m$$

(b) ~~The power emitted by this laser is 5mW. What is the peak electric field in W/cm at~~ What is the radius of curvature of the phase front at this distance?

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$

$$R(z) = 119.7m\left[1 + \left(\frac{24.42m}{119.7m}\right)^2\right] = 124.7m$$

(c) What is the amplitude of the electric field at $r=0$ and $z=0$?

$$P = \frac{1}{2}\frac{E_0^2}{\eta}\left(\frac{\pi W_0^2}{2}\right) \Rightarrow E_0^2 = \frac{4\eta P}{\pi W_0^2}$$

$$\eta = \left(\frac{\mu_0}{\varepsilon_0 n}\right)^{1/2} = \left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2} \sim 377\Omega \qquad P = 1W$$

$$W_0^2 = 2\cdot10^{-3}\,m$$

$$E_0^2 = \frac{4(377\Omega)(1W)}{\pi(2\cdot10^{-3}m)} = 2.4\cdot10^5\ \frac{\cancel{W}^2}{m^2}$$

$$E_0 = 4.9\cdot10^2\,V/m$$

3.4 A 10-W argon ion laser oscillating at 4880 Å has a minimum spot size of 2 mm.

(a) How far will this beam travel before the spot size is 4 mm?

$\cancel{z = 2z_0}$ $\quad W = 2W_0 \quad z_0 = \pi n W_0^2/\lambda_0 \quad W_0 = 2\text{mm}$

$$z_0 = \pi(1.0)(2\cdot10^{-3}\text{m})^2/(4880\cdot10^{-10}\text{m}) = 25.75\text{m}$$

$$W^2 = W_0^2\left[1+\left(\frac{z}{z_0}\right)^2\right] \Rightarrow z^2 = z_0^2\left[\left(\frac{W}{W_0}\right)^2 - 1\right]$$

$$z = z_0\left(\left(\frac{W}{W_0}\right)^2-1\right)^{1/2} = z_0\left(\left(\frac{2W_0}{W_0}\right)^2-1\right)^{1/2} = z_0(4-1)^{1/2} = \sqrt{3}\, z_0$$

$$z = (\sqrt{3})(25.75\text{m}) = 44.6\text{m}$$

b) What fraction of the 10 W is contained in a hole of diameter 2W?

$D = 2W$ hole $\quad 0 \le w \le r$

$$\frac{P(r<w)}{P_{total}} = \frac{\int_0^{2\pi}\int_0^{r=w} I(r)\, r\,dr\,d\phi}{\int_0^{2\pi}\int_0^{\infty} I(r)\, r\,dr\,d\phi} = \frac{\int_0^{2\pi}\int_0^{W} I_0 e^{-2r^2/w^2}\, r\,dr\,d\phi}{\int_0^{2\pi}\int_0^{\infty} e^{-2r^2/w^2}\, r\,dr\,d\phi}$$

$$= \frac{2\pi I_0 (W^2/4)\int_0^{2} e^{-x}dx}{2\pi I_0 (W^2/4)\int_0^{\infty} e^{-x}dx} = \frac{\int_0^2 e^{-x}dx}{\int_0^\infty e^{-x}dx} = \frac{e^{-2}-1}{e^{-\infty}-1} = 1-e^{-2}$$

~ 0.865

Substitute $x = +\frac{2r^2}{W^2}$

$dx = +\frac{4r}{W^2}dr$

c) Express the frequency/wavelength of this laser in eV, nm, µm, ν(Hz), and $\bar{\nu}$(cm^{-1}).

$$\lambda_0 = 4880\,\text{Å} = \frac{4880\,\text{Å}\ \text{nm}}{10\,\text{Å}} = 488\,\text{nm}$$

$\cancel{\lambda_0\ 488\text{nm} = \frac{488\text{nm eV}}{806.549\text{nm}} = 0.605\text{eV}}$

$$\lambda_0\ 488\,\text{nm} = \frac{488\cdot10^{-9}\text{m}\ \mu\text{m}}{10^{-6}\text{m}} = 0.488\,\mu\text{m}$$

$$\nu(\text{Hz}) = \frac{c}{\lambda_0} = \frac{3\cdot10^8\text{m/s}}{4880\cdot10^{-10}\text{m}} = 6.15\cdot10^{14}\,\text{Hz}$$

$$\bar{\nu}(\text{cm}^{-1}) = \frac{1}{\lambda_0} = \frac{10^{-2}\text{m}}{4880\cdot10^{-10}\text{m cm}} = 20491.8\,\text{cm}^{-1}$$

$$1\,\text{Hz} \Leftrightarrow 4.1357\cdot10^{-15}\,\text{eV} = \frac{6.626\cdot10^{-34}\text{Js}}{1.602\cdot10^{-19}\text{J}}\text{eV}$$

$$h\nu(\text{eV}) = \frac{h}{q_e}\nu = \frac{6.626\cdot10^{-34}\text{Js eV}}{1.602\cdot10^{-19}\text{J}}\cdot 6.15\cdot10^{14}\text{Hz} = 2.54\text{eV}$$

d) What is the amplitude of the electric field when w = 1 cm?

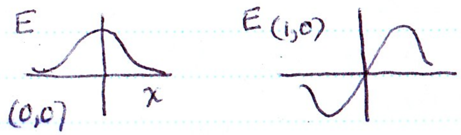
$$E = \sqrt{2\eta I_0} = \sqrt{\frac{2\eta P_0}{(\pi W_0^2/2)}} = \sqrt{\frac{4\eta P_0}{\pi W_0^2}}$$

$$E = \sqrt{\frac{4(377\,\Omega)(10\,\text{W})}{\pi\,(1\cdot10^{-2})^2\text{m}^2}} = 6928.3\,\text{V/m}$$

3.5 Sketch the variation of the intensity with x $(y=0)$ of a beam containing 1W of power in the $TEM_{0,0}$ mode with $\frac{1}{2}$W in $TEM_{1,0}$ mode. $(P_0 = 1.5\,W)$

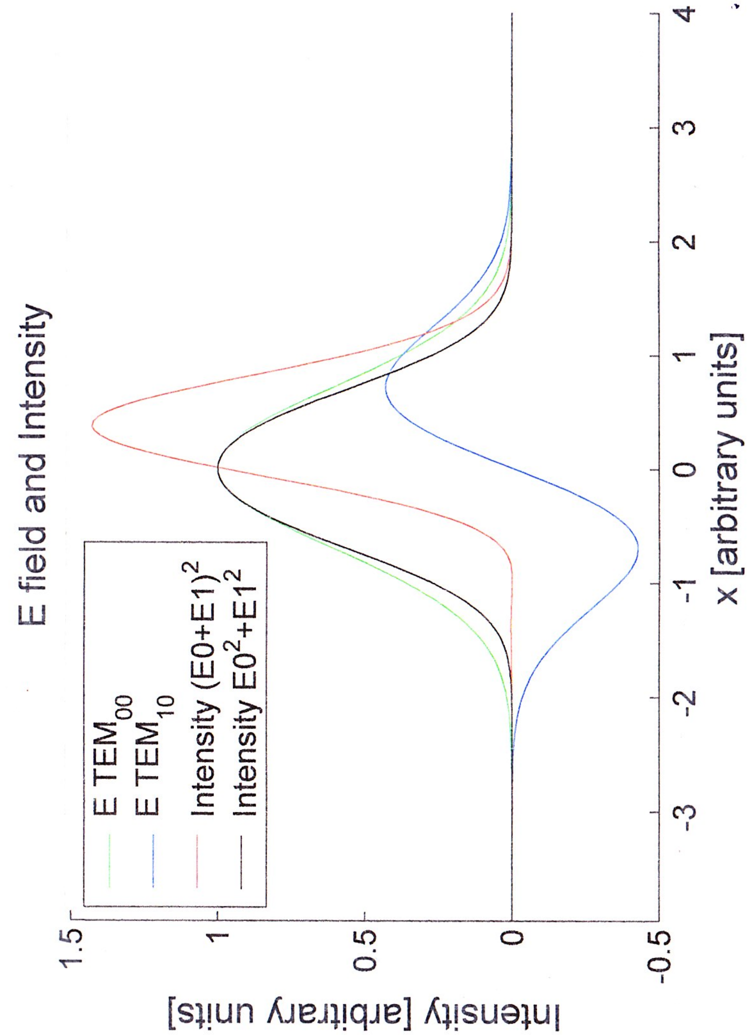
(1) ν, Φ are the same : add $\vec{E}$ and square

(2) $\Phi_1 \neq \Phi_2$: add intensities



C:\MATLAB6p5p2\work\Lasers_Ch3_Pr5.m
April 23, 2009

```
function Lasers_Ch3_Pr5
textsize = 10;titlesize = 14;axissize = 12;
nx=200;gain=8.0/nx;offset=-gain*nx/2.0;
x = zeros(nx);for i = 1:nx;x(i) = i*gain+offset;end;
E00 = exp(-x.^2);
E10 = sqrt(.5).*sqrt(2.0).*x.*exp(-x.^2)
axes('fontsize',axissize,'position',[.15 .19 .81 .7]);
axis([x(1) x(nx) -0.5 1.5]);
ifigure = 1;figure(ifigure);cla;
title(['E field and Intensity'],'fontsize',titlesize);
ylabel('Intensity [arbitrary units]','fontsize',titlesize);
xlabel('x [arbitrary units]','fontsize',titlesize);hold on;
h1=plot(x,E00,'color',[0 .7 0]);
h1=plot(x,E10,'b',x,(E00+E10).^2,'r',x,E00.^2+E10.^2,'k');
h=legend(['E TEM_0_0          ';'E TEM_1_0          ';
    'Intensity (E0+E1)^2';'Intensity E0^2+E1^2'],2);
lines = findobj(get(h,'children'),'type','line');
set(lines(1),'color',[0 .7  0]);
set(lines(2),'color',[0  0 0]);
set(lines(4),'color',[.7 0  0]);
set(lines(6),'color',[0 0 .7]);
```



3.6 Consider a linear combination of two equal amplitude $TEM_{m,p}$ modes given by:

$$\vec{E} = E_0 \{ (TEM_{1,0})\hat{a}_y \pm j (TEM_{0,1})\hat{a}_x \}$$

a) Sketch the 'dot' pattern or equal intensity contours for each component. Indicate the field direction

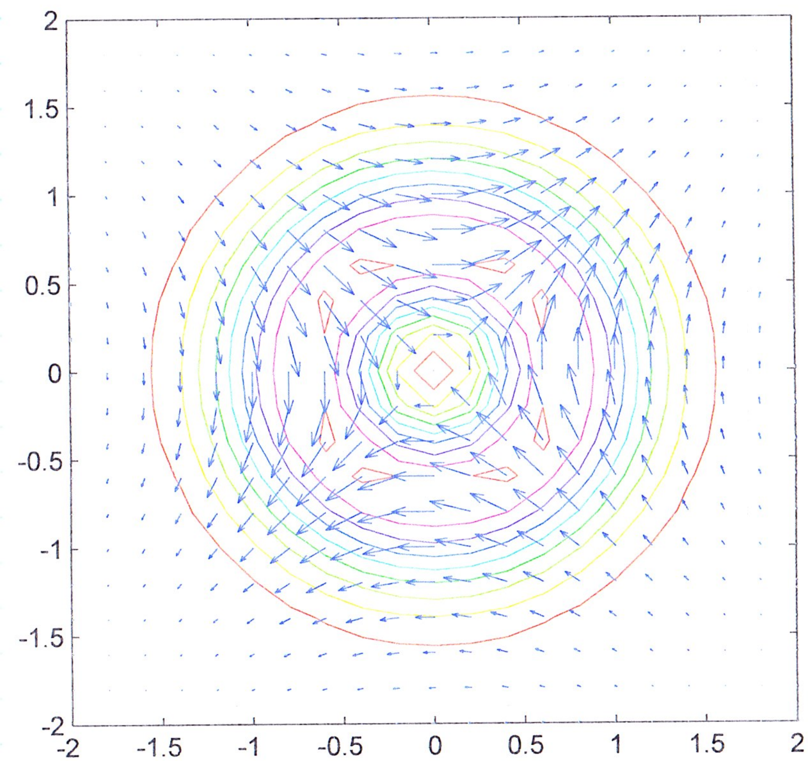
b) Sketch the pattern for the linear combination.

c) Label the positions where the intensity is a maximum and minimum.

C:\MATLAB6p5p2\work\Lasers_Ch3_Pr6.m
April 23, 2009

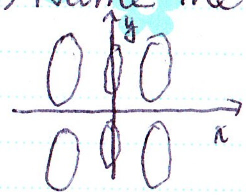
```
function Lasers_Ch3_Pr6
figure(1);true;
axis([-2 2 -2 2]);
[X,Y] = meshgrid(-2:.2:2);
R2 = X.^2+Y.^2
DX=Y.*exp(-R2);
DY=X.*exp(-R2);
Z = DX.^2+DY.^2;
contour(X,Y,Z)
hold on
quiver(X,Y,DX,DY)
colormap hsv
grid off
hold off
```

Donut Mode $TEM^*_{0,1}$



3.7 The intensity of a laser has the following visual appearance when projected on a surface.

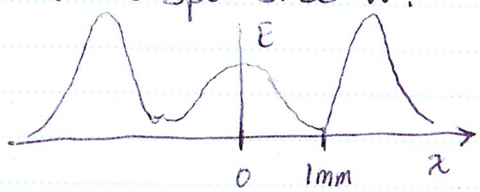
a) Name the mode



$m = 2$ $\quad$ $TEM_{2,1}$

$p = 1$

b) A plot of the intensity of another mode as a function of x (for y=0) is shown at the right. The variation with respect to y is a simple bell-shape curve. What is the spot size w?



$(2u^2 - 1) = 0 \Rightarrow E = E_0 H_2(u) \left(\frac{W_0}{W(z)} e^{-\frac{x^2+y^2}{w^2}}\right) e^{-jkz} e^{-j\theta_r} = 0$

$u^2 = 1/2$

$u = \pm 1/\sqrt{2}$

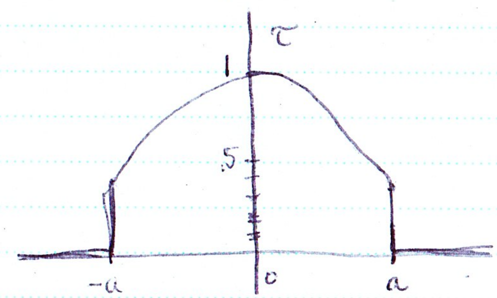
$u = \frac{\sqrt{2}\, x}{W(z)} = \frac{\pm 1}{\sqrt{2}} \Rightarrow x\Big|_{E=0} = \frac{W(z)}{2} \Rightarrow W_0 = 2mm$

3.8 Suppose a $TEM_{m,p}$ mode impinged on a perfectly absorbing plate with a hole of radius a centered on the axis of the beam. Plot the transmission coefficient of this hole as a function of the ratio a/w for the (0,0), (0,1) and (1,1) mode assuming that the fields are not affected by the plate.

The reflection coefficient $r = \left|\frac{E}{E_0}\right|$ on a miror becomes the ~~trans~~ aperture transmission coefficient in the equivalent lensguide model.

$t = \left|\frac{E}{E_0}\right| = \frac{W_0}{w(z)} \exp\left(-(r/w)^2\right) \qquad w \to w_0$

$t(a/w) = \exp\left(-(r/a)^2\right)$



3.9 Show that the Hermite Gaussian beam modes are orthogonal in the following sense:

$$\mathrm{Re}\int(\vec{E}_{m,n} \times \vec{H}^*_{p,q}) \cdot d\vec{S} = 0$$

$d\vec{S} = dx\,dy \qquad \vec{S} = \vec{E} \times \vec{H}^* \qquad d\vec{S} = dx\,dy\,\hat{z}$

$\vec{E}_{m,n} \propto H_m(x) H_n(y) \hat{x} \qquad \vec{H}^*_{p,q} \propto H_p(x) H_q(y) \hat{y}$

$$\vec{S} = \vec{E}_{mn} \times \vec{H}^*_{p,q} \propto H_m(x) H_p(x) H_n(y) H_p(y)\,\hat{z}$$

Hermite polynomials are orthogonal!

3.10 Repeat the analysis from 3.2.6 to 3.3.14 for the case where the index of refraction is non-uniform and is given by: $n(r) = n_0(1 - r^2/2\ell^2) \qquad r^2 = x^2 + y^2$

$$\nabla_t^2 E + \partial^2 E/\partial z^2 + (\omega^2/c^2) n^2 E = 0$$

$$\nabla_t^2 E + \partial^2 E/\partial z^2 + \frac{\omega^2}{c^2} n_0^2 \left(1 - \frac{r^2}{\ell^2} + \frac{r^4}{4\ell^4}\right) E = 0$$

trial solution: $E = E_0 \psi(r) e^{-jk_0 z} e^{\alpha z}$

$$E_0 \nabla_t^2 \psi e^{-jk_0 z} + E_0\left(-k_0^2 \psi - 2jk_0 \frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2}\right) e^{-jkz}$$

$$+ \frac{\omega^2}{c^2} n_0^2 \left(1 - \frac{r^2}{\ell^2} + \frac{r^4}{4\ell^4}\right) E_0 \psi e^{-jkz} = 0$$

$$\nabla_t^2 \psi + \left(-k_0^2 \psi - 2jk_0 \frac{\partial \psi}{\partial z} + \cancel{\frac{\partial^2 \psi}{\partial z^2}}\right)$$ neglect because k is large

$$+ \frac{\omega^2}{c^2} n_0^2 \left(1 - \frac{r^2}{\ell^2} + \frac{r^4}{4\ell^4}\right)\psi = 0$$

since ψ isn't a function of z

$$\nabla_t^2 \psi + \frac{\omega^2}{c^2} n_0^2 \left(1 - \frac{k_0^2 c^2}{n_0^2 \omega^2} - \frac{r^2}{\ell^2} + \frac{r^4}{4\ell^4}\right)\psi = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r}\right) - j2k_0 \frac{\partial \psi}{\partial z} + \frac{\omega^2}{c^2} n_0^2 \left(1 - \frac{k_0^2 c^2}{n_0^2 \omega^2} - \frac{r^2}{\ell^2} + \cancel{\frac{r^4}{4\ell^4}}^{\,0}\right)\psi = 0$$

3.10

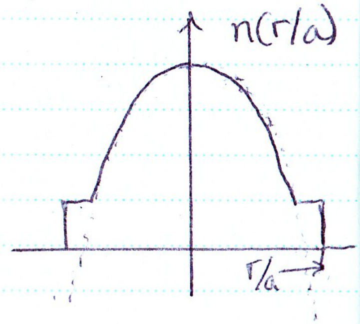
Book on-line Optics, Lights + Lasers By Dieter Meschede

Graded Index Fiber

"quadratic index media"

n is a function of normalized radius r/a

$n(\rho)=n_1[1-\Delta(r/a)^2]$ $\quad \Delta \ll 1$



Solutions to Helmholtz eq.

$$(\nabla^2+k^2)\vec{E}(\vec{r})=0$$

trial soln: $E(x,y,z)=A(x,y)e^{-i\beta z}$

modified equation:

$$\{\nabla_\perp + n_1^2k^2 - 2n_1^2k^2\Delta[(\tfrac{x}{a})^2+(\tfrac{y}{a})^2] - \beta^2\}A(x,y)=0$$

using

$$\{n_1k[1-\Delta(r/a)^2]\}^2 \simeq (n_1k)^2[1-2\Delta(r/a)+\dots]$$

assume that the transverse distribution are modified Gaussian functions:

$$A(x,y)=F(x)e^{-(x/x_0)^2}G(y)e^{-(y/y_0)^2}$$

With this ansatz we find:

$$(F''-\tfrac{4x}{x_0^2}F'-\tfrac{2}{x_0^2}F)G+(G''-\tfrac{4y}{y_0^2}G'-\tfrac{2}{y_0^2}G)F$$

$$+n_1^2k^2FG+\Big[\big(\tfrac{4}{x_0^4}-\tfrac{2n_1^2k^2\Delta}{a^2}\big)x^2$$

$$+\big(\tfrac{4}{y_0^4}-\tfrac{2n_1^2k^2\Delta}{a^2}\big)y^2\Big]FG-\beta^2FG=0$$

eliminate the quadratic term via:

$$kx_0=ky_0=(ka)^{1/2}/(2n_1^2\Delta)^{1/4} \gg 1$$

Subsitute $\sqrt{2}\,\frac{x}{x_0}\to u$ and $\sqrt{2}\,\frac{y}{y_0}\to v$ transforms to the Hermite differential equation:

$$2(F''-2uF'+2mF)G+2(G''-2vG'+2nG)F$$

$$+[n_1^2k^2x_0^2-\beta^2x_0^2-4(m+n+1)]FG=0$$

The upper row terms vanish with Hermite polynomials $H_{m,n}$. The propagation constant becomes:

$$\beta_{m,n}(\omega)=n_{eff}\frac{\omega}{c}=\frac{n_1\omega}{c}\sqrt{1-\frac{4\sqrt{2\Delta}(m+n+1)}{n_1ka}}$$

"In contrast to the Gaussian modes, the mode diameters (x_0,y_0) do not change."

"The simplified GRIN fiber illustrates that multimode fibers characterized by a frequency dependent index of refraction show 'mode dispersion' in addition to 'material dispersion.'" This influences the form of pulses because individual partial modes have different propagation velocities.

3.11 The same arguments advanced for the derivation of (3.2.4) can be used for the magnetic field intensity H. Check the accuracy of this equation by considering a dominant $TE_{1,0}$ mode in a rectangular wave guide of width a and height b and computing the ratio of H_z/H_x

In a linear, source free, isotropic, homogeneous region Maxwell's curl equations are in phasor form are:

$$\vec{\nabla}\times\vec{E} = -j\omega\mu\vec{H} \quad \vec{\nabla}\times\vec{H} = j\omega\varepsilon\vec{E} \quad \text{let } \vec{E}_{\phi} = E_y\hat{y}$$

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \cancel{E_x} & E_y & 0 \end{vmatrix} = -\hat{k}\,\partial/\partial z\, E_y + \hat{j}\frac{\partial E_y}{\partial x} = -j\omega\mu\vec{H}$$

$$\therefore \vec{H} = i\left(\frac{1}{\omega\mu}\right)\left[\hat{j}\frac{\partial E_y}{\partial x} - \hat{k}\frac{\partial E_y}{\partial z}\right]$$

$$\vec{E} = E_{10} H_m\left(\frac{\sqrt{2}x}{w(z)}\right) H_p\left[\frac{\sqrt{2}y}{w(z)}\right]\frac{w_0}{w_0(z)} e^{-(x^2+y^2)/w^2(z)} e^{-j[kz-(1+m+p)\tan^{-1} z/z_0}$$

$$\cdot e^{-j\,kr^2/2R(z)}\Big]$$

$$\vec{E}_{10} = E_0\left(\frac{\sqrt{2}x}{w(z)}\right)(1)\frac{w_0}{w_0(z)} e^{-(x^2+y^2)/w^2(z)} e^{-j[kz-2\tan^{-1} z/z_0]} e^{-jkr^2/2R}$$

3.12 The news media has shown the astronauts placing laser retroreflectors on the moon. Use the expansion law for Gaussian beams to predict the diameter of a laser beam when it hits the moon. Use $\lambda_0 = 6943 Å$
Consider two cases:

a) a laser rod of 2cm diameter $\rightarrow w_0 = 1cm$

take $n = 1$

$\frac{\theta/2}{R_{ME}} \rfloor r$

$(\theta/2) = \lambda_0/\pi n w_0 = 6943 \cdot 10^{-10} m/(\pi \cdot 2 \cdot 10^{-2} m)$

$(\theta/2) = 1.1 \cdot 10^{-5}$ rad $= (6.33 \cdot 10^{-4})^\circ$

$D = 2r \quad \frac{r}{R_{ME}} \sim \frac{\theta}{2} \quad D \sim \theta R_{ME} \sim \frac{2\lambda_0}{\pi n w_0} R_{ME}$

$$D \sim \frac{2(6943 \cdot 10^{-10} m)(38\cancel{440} \cdot 10^3 m)}{\pi (\cancel{1} 2 \cdot 10^{-2} m)}$$ ← rough distance

~~$D \sim 850 m$~~

$R'_{ME} = R_{ME} - R_m - R_E = 384403 km - 1737 km - 6371 km = 376295 km$

$D = 2r \sim 2\left(\frac{\theta}{2}\right) R'_{ME} = \frac{2\lambda_0}{\pi n w_0} R'_{ME}$

$$D \sim \frac{(6943 \cdot 10^{-10} m)(376295 \cdot 10^3 m) 2}{\pi (2 \cdot 10^{-2} m)}$$

$D \sim$ ~~4.2 km~~ 8.3 km if $w_0 \sim 2cm$
but $w_0 \sim 1cm$

$D = 2r \quad \frac{r}{R'_{ME}} = \tan(\theta/2)$

$D = 2 R'_{ME} \tan(\theta/2)$

$D = 2 R'_{ME} \tan(\lambda_0/\pi n w_0)$

$D = 2(384403 \cdot 10^3 m) \tan(6943 \cdot 10^{-10} m/\pi \, 1 \cdot 10^{-2} m)$

$D \sim$ ~~8.5 km~~ 16.99 km

b) $w_0 = 1m \quad \lambda_0 = 6943 \cdot 10^{-10} m$

$D = 2 R'_{ME} \tan(\lambda_0/\pi n w_0)$

$D = 2(384403 km) \tan(6943 \cdot 10^{-10} m/\pi \, 1m)$

$D =$ ~~0.085 km~~ $=$ ~~85 m~~ $0.17 km = 169.9 m$

c) Eye damage intensities are in the range of $10 \mu W/cm^2$. If the laser on earth produced a pulse power of 10MW was there danger to the astronauts from the optical radiation.

$$I_a = \frac{10 \cdot 10^6 W}{\pi (\cancel{8.5}\ 16.99 \cdot 10^3 m)^2} \left(\frac{m}{10^2 cm}\right)^2 \left(\frac{10^6 \mu W}{W}\right) = 1.1 \mu W/cm^2$$

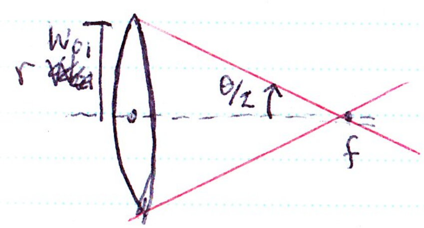
$$I_b = \frac{10 \cdot 10^6 W}{\pi (\cancel{85}\ 170 m)^2} \left(\frac{m}{10^2 cm}\right)^2 \left(10^6 \frac{\mu W}{W}\right) = \cancel{44{,}059}\ 11{,}014 \mu W/cm^2$$

in the second case, yes.

3.13 Verify the ABCD law for a continuous lens by starting with (3.6.16b) and following the analysis of Sec. 3.1 through 3.3.

Essentially this is the same as problem 3.10.

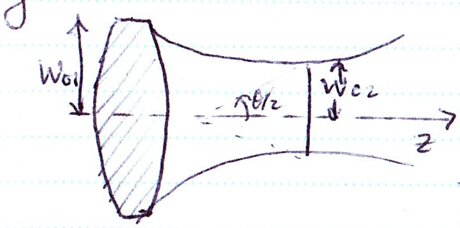
3.14 A convenient, if oversimplified, definition of a focal length of a lens is that it converges a parallel beam of light to a point. But if the spot size of a point were zero, the expansion of the beam would be infinitely fast and by symmetry would also correspond to its convergence, both statements being obvious contradictions. Use a simple geometric argument based on the convergence (and expansion) to estimate ~~a simple geometric argument based on the convergence~~ the minimum spot size in the focal region of the lens. Compare to exact answer.



$\tan\left(\frac{\theta}{2}\right) = \frac{r}{f}$

$r = w_{01}$ figure 3.6

Fig. 3.6



$\theta/2 = \lambda_0/\pi n w_{02}$ (3.4.2)

$\Rightarrow w_{01} = f\tan\frac{\theta}{2} = f\tan\left(\frac{\lambda_0}{\pi n w_{02}}\right)$

$w_{02}^{-1} = \frac{\pi n}{\lambda_0}\tan^{-1}\left(\frac{w_{01}}{f}\right)$

$\boxed{\text{pt:} \quad w_{02} = \frac{\lambda_0}{\pi n}\tan^{-1}\left(\frac{f}{w_{01}}\right)}$

The minimum spot size ~~in the exact~~ solution is not at the focal length.

$z_m = f/(1+(f/z_{01})^2)$

3.6.12 $\rightarrow$ $\boxed{w_{02} \sim \frac{\lambda_0 f}{\pi w_{01}}}$ same in the small angle approximation

3.15 Suppose that a Gaussian beam of $w = 2$ cm and a planar wave front impinges on a lens of focal length $f = 4$ cm ($\lambda_0 = 1.0\ \mu m$).

a) If $z=0$ is the location of the lens, where does the output beam reach its minimum spot size?

$$z_m = \frac{f}{1 + (f/z_0)^2}$$

$$z_{01} = \frac{\pi W_{01}^2}{\lambda_0} = \frac{\pi (2 \cdot 10^{-2} m)^2}{1 \cdot 10^{-6} m} = 1.257 \cdot 10^3 m$$

$$z_m = \frac{4 \cdot 10^{-2} m}{1 + (4 \cdot 10^{-2} m / 1.257 \cdot 10^3 m)^2} \sim 0.04 m \sim f$$

b) What is the far-field expansion angle?

not sure which is correct!

$$\tan \theta/2 = W_{01}/f = 2/4 \Rightarrow \theta/2 = .464 \text{ rad}$$

$$\theta/2 = \frac{\lambda_0}{\pi n W_{02}} = \frac{1 \cdot 10^{-6} m}{\pi (2 \cdot 10^{-2} m)} = 1.59 \cdot 10^{-5} \text{ rad}$$

3.16 Repeat the analysis of Sec 3.1 → Sec 3.3 for a medium in which the dielectric constant is complex and depends on r in the following manner:

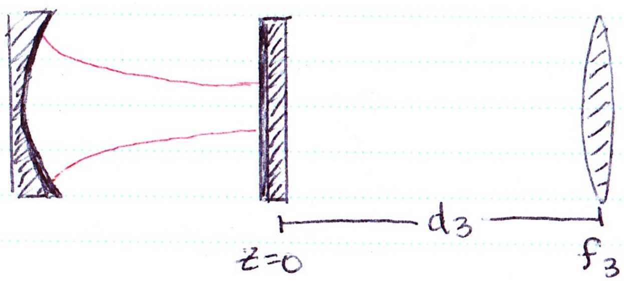
$$\varepsilon(r) = \varepsilon_0 [\varepsilon' - j\varepsilon''(1 - r^2/w_0^2)]$$

The term ε'' can be positive or negative corresponding to gain or loss, but in any case, it is much less than ε' and the scale length w_0 is much larger than r.

Note: $n^2 = \frac{\varepsilon}{\varepsilon_0} = \varepsilon' - j\varepsilon''(1 - r^2/w_0^2)$

has same form as problem 3.10!

3.17 The laser cavity shown below produces a $TEM_{0,0}$ mode with $z=0$ located at the flat mirror and its output impinges on a lens of focal length f_3.
Assume w_0 is known (0.5 mm)
$\lambda_0 = 6328$ Å $\quad d_3 = 1m \quad f_3 = 0.25m$



a) what are the spot size and radius of the curvature of the wave impinging on f_3?

Spot size $w(z)$ given by equation 3.3.10

$$w^2(z) = w_0^2\left[1 + \left(\frac{\lambda_0 z}{\pi n w_0^2}\right)^2\right]$$

$z = d_3 = 1m \quad \lambda_0 = 6328 \cdot 10^{-10} m$
$w_0 = .5 \cdot 10^{-3} m \quad n = 1$

$$w^2(d_3) = (.5 \cdot 10^{-3} m)^2\left[1 + \left(\frac{(6328 \cdot 10^{-10} m)(1m)}{\pi(1)(0.5 \cdot 10^{-3} m)^2}\right)^2\right]$$

$$w^2(d_3) = 4.1 \cdot 10^{-7} m^2$$

$$w(d_3) = 6.42 \cdot 10^{-4} m = 0.642 mm$$

The radius of curvature is given by 3.3.11

$$R(z) = z\left[1 + \left(\frac{\pi n w_0^2}{\lambda_0 z}\right)^2\right]$$

$$R(d_3) = (1m)\left[1 + \left(\frac{\pi(.5 \cdot 10^{-3} m)^2}{(6328 \cdot 10^{-10} m)(1m)}\right)^2\right]$$

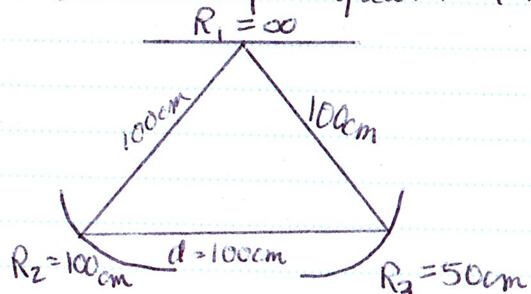
$$R(d_3) = 2.54 m$$

b) What is the radius of curvature after passage through f_3?

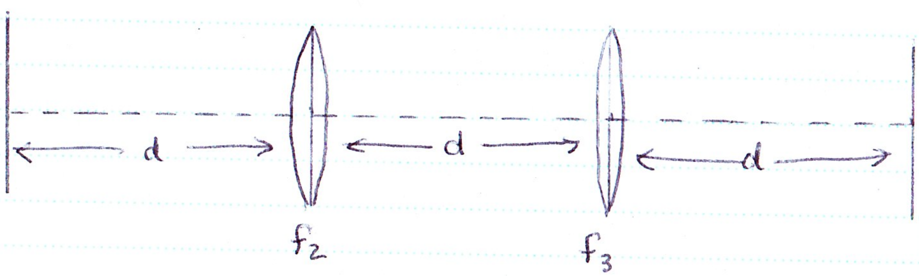
$$R = 2f = 2f_3 = 2(.25m) = .5m$$

3.18 Is the cavity shown below stable?

a) construct unit cell start at flat mirror
b) find ABCD matrix
c) apply stability criteria
d) What are the circumstances under which the quantity [AD-BC] can be different from 1? Why is AD-BC always equal to 1 for a cavity?



unit cell



$d = 100cm$

$f_2 = \frac{R_2}{2} = 50cm$

$f_3 = \frac{R_3}{2} = 25cm$

$$T = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & d \\ -1/f_3 & (1-d/f_3) \end{bmatrix}\begin{bmatrix} 1 & d \\ -1/f_2 & (1-d/f_2) \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1-d/f_2 & d+d-d/f_2 \\ -1/f_3 - 1/f_2(1-d/f_3) & -d/f_3 + (1-d/f_3)(1-d/f_2) \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} (1-d/f_2) & d(2-1/f_2) \\ -[(1/f_3)+(1/f_2)(1-d/f_3)] & \{-d/f_3 + (1-d/f_3)(1-d/f_2)\} \end{bmatrix}$$

$$T = \begin{bmatrix} (1-d/f_2) - d(1/f_3) - d(1/f_2)(1-d/f_3) & d(2-1/f_2) + d(1-d/f_3)(1-d/f_2) \\ -1/f_3 - 1/f_2(1-d/f_3) & -d/f_3 + (1-d/f_3)(1-d/f_2) \end{bmatrix}$$

$$T = \begin{bmatrix} 1-\frac{2d}{f_2} - \frac{d}{f_3} + \frac{d^2}{f_2 f_3} & d(2-1/f_2+1-d/f_3-d/f_2+d^2/f_2 f_3) \\ -1/f_3 - 1/f_2 + d/f_2 f_3 & 1-d/f_2 - 2d/f_3 + d^2/f_2 f_3 \end{bmatrix}$$

$$T = \begin{bmatrix} 1-\frac{2d}{f_2} - \frac{d}{f_3} + \frac{d^2}{f_2 f_3} & d(3-2/f_2 - 1/f_3 + d/f_2 f_3) \\ -1/f_3 - 1/f_2 + d/f_2 f_3 & 1-d/f_2 - 2d/f_3 + d^2/f_2 f_2 \end{bmatrix}$$

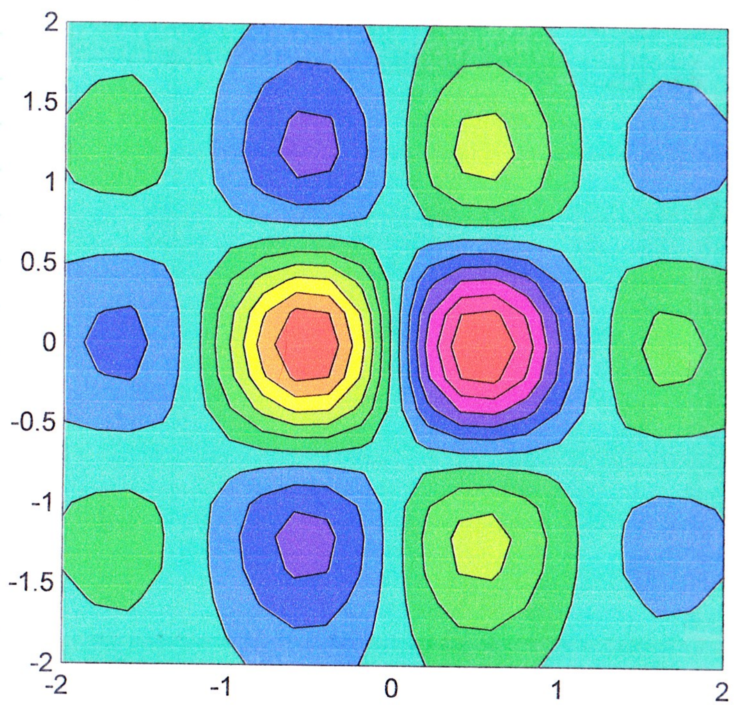
$d = 100$ cm $\quad f_2 = 50$ cm $\quad f_3 = 25$ cm

$$T = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$$

$$S = \frac{A+D+2}{4} = \frac{1}{2} \Rightarrow \text{Stable}$$

3.20 Sketch the dot pattern (contours of x/w and y/w) for $TEM_{3,2}$

$H_2 = 4\xi^2 - 2$ $\quad H_3 = 8\xi^3 - 12\xi$



C:\MATLAB6p5p2\work\Lasers_Ch3_Pr20.m
April 29, 2009

```
function Lasers_Ch3_Pr20
figure(1);true;
axis([-2 2 -2 2]);
[X,Y] = meshgrid(-2:.15:2);
R2 = X.^2+Y.^2
xmode = 3;
ymode = 2;
switch xmode
    case 0
        Hx = 1.0;
    case 1
        Hx = 2.0*X;
    case 2
        Hx = (4.0*X.^2-2.0);
    case 3
        Hx = (8.0*X.^3-12.0*X);
    otherwise
        Hx = 1.0;
end
switch ymode
    case 0
        Hy = 1.0;
    case 1
        Hy = 2.0*Y;
    case 2
        Hy = (4.0*Y.^2-2.0);
    case 3
        Hy = (8.0*Y.^3-12.0*Y);
    otherwise
        Hy = 1.0;
end
Umn = Hx.*Hy.*exp(-R2);
contourf(X,Y,Umn,12)
surface(X,Y,Umn)
view(-35,55)
hold on
colormap hsv
grid off
hold off
```