Chapter 1: Review of Electromagnetic Theory

1.1 Why is the factor 2 present in the expression for the Poynting vector?

$$\vec{S} = \frac{\vec{E} \times \vec{H}^*}{2}$$

With the factor of ½ the time average of the Poynting vector $\langle \vec{S} \rangle$ is equal to the intensity (irradiance).

1.2 Assume that the electromagnetic fields vary as $\exp(-j\vec{k}\cdot\vec{r})$ and use the rules for curl and gradient and divergence to derive the algebraic form of maxwell's equations (1.4.2).

$$\vec{\nabla} \times \vec{E} = -j\omega\mu_0 \vec{H} \qquad \vec{E} = E_{ox}e^{-j\vec{k}\cdot\vec{r}}\hat{i} + E_{oy}e^{-j\vec{k}\cdot\vec{r}}\hat{j} + E_{oz}e^{-j\vec{k}\cdot\vec{r}}\hat{k}$$
$$= \vec{E}_x\hat{i} + \vec{E}_y\hat{j} + \vec{E}_z\hat{k}$$

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \vec{E}_x & \vec{E}_y & \vec{E}_z \end{vmatrix}$$

$$\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$$

$$\frac{\partial}{\partial x} E_0 e^{-j\vec{k}\cdot\vec{r}} = -jk_x E_0 e^{-j\vec{k}\cdot\vec{r}}$$
$$\frac{\partial}{\partial y} E_0 e^{-j\vec{k}\cdot\vec{r}} = -jk_y E_0 e^{-j\vec{k}\cdot\vec{r}}$$
$$\frac{\partial}{\partial z} E_0 e^{-j\vec{k}\cdot\vec{r}} = -jk_z E_0 e^{-j\vec{k}\cdot r}$$

$$\vec{\nabla} \times \vec{E} = \left(\frac{\partial}{\partial y}\vec{E}_z - \frac{\partial}{\partial z}\vec{E}_y\right)\hat{i} + \left(\partial/\partial z\,\vec{E}_x - \partial/\partial x\,\vec{E}_z\right)\hat{j} + \left(\partial/\partial x\,\vec{E}_y - \partial/\partial y\,\vec{E}_x\right)\hat{k}$$

$$\vec{\nabla} \times \vec{E} = \left(-jk_y E_{oz}e^{-j\vec{k}\cdot\vec{r}} + jk_z E_{oy}e^{-j\vec{k}\cdot\vec{r}}\right)\hat{i} + \left(-jk_z E_{ox}e^{-j\vec{k}\cdot\vec{r}} + jk_x E_{oz}e^{-j\vec{k}\cdot\vec{r}}\right)\hat{j} + \left(-jk_x E_{oy}e^{-j\vec{k}\cdot\vec{r}} + jk_y E_{ox}e^{-j\vec{k}\cdot\vec{r}}\right)\hat{k}$$

$$\vec{k} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ k_x & k_y & k_z \\ \vec{E}_x & \vec{E}_y & \vec{E}_z \end{vmatrix} = (k_y\vec{E}_z - k_z\vec{E}_y)\hat{i} + (k_z\vec{E}_x - k_x\vec{E}_z)\hat{j} + (k_x\vec{E}_y - k_y\vec{E}_x)\hat{k} \Rightarrow \frac{1}{j}\vec{\nabla} \times \vec{E} = -\vec{k} \times \vec{E}$$

$$\therefore \cancel{\vec{\nabla} \times \vec{E}} = \quad \vec{k} \times \vec{E} = +\omega\mu_0\vec{H}$$

Similarly, by substition of variables $\vec{H} \leftrightarrow \vec{E}$

$$-j\vec{\nabla}\times\vec{H} = \vec{k}\times\vec{H}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + j\omega\varepsilon_0\vec{E} + j\omega\vec{P}$$

in free space $\vec{J} = 0$ and $\vec{P} = 0 \Rightarrow \vec{\nabla}\times\vec{H} = j\omega\varepsilon_0\vec{E}$

$$\therefore \; -j(\vec{\nabla}\times\vec{H}) = -j(j\omega\varepsilon_0\vec{E})$$

$$\vec{k}\times\vec{H} = -\omega\varepsilon_0\vec{E}$$

1.3 The algebraic forms of Maxwell's Equations for a linear homogeneous anisotropic medium are:

$$\vec{k}\times\vec{H} = -\omega\vec{D}$$
$$\vec{k}\times\vec{E} = \omega\vec{B}$$

where $\vec{B}$ is related to $\vec{H}$ and $\vec{D}$ to $\vec{E}$ by

$$\vec{B} = \mu_0(\vec{H} + \vec{M})$$
$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

For many materials, the polarization vector $\vec{P}$ is not collinear with $\vec{E}$, hence $\vec{D}$ is not collinear with $\vec{E}$ either. The same comments apply to $\vec{B}$, $\vec{M}$, and $\vec{H}$. Assume a dielectric medium with $\vec{M} = 0$ but with no restrictions placed on $\vec{D}$ and $\vec{E}$.

(a) Show that $\vec{k}\cdot\vec{D} \equiv 0$.

$\cancel{\vec{k}\times\vec{E} = \vec{k}\times(\tfrac{1}{\varepsilon_0}\vec{D} - \tfrac{1}{\varepsilon_0}\vec{P}) \therefore \vec{k}\times\vec{D} \propto \omega\vec{B}}$

$\vec{k}\times\vec{H} = -\omega\vec{D} \;\therefore\; \vec{k}\perp\vec{D} \;\therefore\; \vec{k}\cdot\vec{D} = 0$

(b) Show that the wave vector $\vec{k}$ always points in the direction of $\vec{D}\times\vec{B}$.

$-\omega^2(\vec{D}\times\vec{B}) = (\vec{k}\times\vec{H})\times(\vec{k}\times\vec{E}) =$

$\vec{k}\{\vec{k}\cdot(\vec{H}\times\vec{E})\} - \vec{E}\{\underbrace{\vec{k}\cdot(\vec{H}\times\vec{k})}_{\to 0 \text{ because } \perp k\cdot k = 0}\}$

$= \vec{k}\{\vec{k}\cdot -(\vec{E}\times\vec{H})\} = -\vec{k}\{\vec{k}\cdot\vec{S}\}$

$\therefore \; \vec{k} = \dfrac{\omega^2(\vec{D}\times\vec{B})}{\{\vec{k}\cdot\vec{S}\}}$

using vector identity 20.21a in Schaum's

(c) Show that the ~~wave vector $\vec{k}$ always points in the direction~~ of amplitude of the wave vector $\vec{k}$ is given by

$$k^2 = \omega^2 \mu \frac{\vec{D}\cdot\vec{D}}{\vec{E}\cdot\vec{D}}$$

$$\vec{k}\times\vec{H} = -\omega\vec{D}$$

$$(\vec{k}\times\vec{H})\cdot(\vec{k}\times\vec{H}) = \omega^2\vec{D}\cdot\vec{D}$$

$$(\vec{k}\cdot\vec{k})\vec{H}\cdot\vec{H} - \underbrace{(\vec{k}\cdot\vec{H})}_{0 \text{ because } \vec{k}\perp\vec{H}}(\vec{H}\cdot\vec{k}) = \omega^2\vec{D}\cdot\vec{D}$$ vector identity 20.20

$$k^2 = \frac{\omega^2\vec{D}\cdot\vec{D}}{\vec{H}\cdot\vec{H}}$$

$$\vec{k}\times\vec{E} = \omega\vec{B}$$ same approach

$$\Rightarrow k^2 = \frac{\omega^2\vec{B}\cdot\vec{B}}{\vec{E}\cdot\vec{E}}$$

(d) Show that the Poynting vector $\vec{S} = (\vec{E}\times\vec{H}^*)/2$ can point in a direction other than that of the wave vector $\vec{k}$.

$$\vec{S} = \frac{\vec{E}\times\vec{H}^*}{2} = \frac{\frac{1}{\varepsilon_0}(\vec{D}-\vec{P})\times H^*}{2} = \frac{\frac{1}{\varepsilon_0}\vec{D}\times\vec{H}^*}{2} + \frac{\frac{1}{\varepsilon_0}\vec{P}\times\vec{H}^*}{2}$$

$$\vec{S} = \frac{\frac{-1}{\omega\varepsilon_0}(\vec{k}\times\vec{H})\times\vec{H}^*}{2} - \frac{\frac{1}{\varepsilon_0}\vec{P}\times\vec{H}}{2}$$

$$\vec{S} = -\frac{\vec{H}(\vec{k}\cdot\vec{H}^*) + \vec{k}(\vec{H}\cdot\vec{H}^*)}{\omega\varepsilon_0 2} - \frac{\vec{P}\times\vec{H}}{2\varepsilon_0}$$

$\vec{k}\cdot\vec{B} = 0$ because $\vec{k}\perp\vec{B}$ $\quad \vec{k}\perp\vec{D}$

$$\vec{S} = -\frac{(\vec{k}\cdot\vec{H}^*)\vec{H}}{\omega\varepsilon_0 2} + \frac{(\vec{H}\cdot\vec{H}^*)\vec{k}}{2\varepsilon_0\omega} - \frac{\vec{P}\times(\frac{1}{\mu_0}\vec{B}-\vec{M})}{2\varepsilon_0}$$

$$\vec{S} = \frac{(\vec{H}\cdot\vec{H}^*)\vec{k}}{2\varepsilon_0\omega} - \frac{(\vec{k}\cdot\vec{H}^*)\vec{H}}{2\varepsilon_0\omega} - \frac{\vec{P}\times(\vec{k}\times\vec{E})}{2\varepsilon_0\mu_0} + \frac{\vec{P}\times\vec{M}}{2\varepsilon_0}$$

$-(\vec{k}\times\vec{P}\times\vec{E})$

$$\vec{S} = \frac{(\vec{H}\cdot\vec{H}^*)\vec{k}}{2\varepsilon_0\omega} - \frac{(\vec{k}\cdot\vec{H}^*)\vec{H}}{2\varepsilon_0\omega} + \frac{\vec{k}(\vec{P}\cdot\vec{E})}{2\varepsilon_0\mu_0} - \frac{\vec{P}(\vec{k}\cdot\vec{E})}{2\varepsilon_0\mu_0} + \frac{\vec{P}\times\vec{M}}{2\varepsilon_0}$$

$$\vec{S} = \frac{H^2\vec{k}}{2\varepsilon_0\omega} - \frac{[\vec{k}\cdot(\frac{1}{\mu}\vec{B}-\vec{M})]\vec{H}}{2\varepsilon_0\omega} + \frac{(\vec{P}\cdot\vec{E})\vec{k}}{2\varepsilon_0\mu_0} + \frac{\vec{P}(\vec{k}\cdot(\frac{1}{\varepsilon}\vec{P}-\vec{D}))}{2\varepsilon_0\mu_0} + \frac{\vec{P}\times\vec{M}}{2\varepsilon_0}$$

$$\vec{S} = \left(\frac{H^2}{2\varepsilon_0\omega} + \frac{\vec{P}\cdot\vec{E}}{2\varepsilon_0\mu_0}\right)\vec{k} - \left(\frac{\vec{k}\cdot\vec{M}}{2\varepsilon_0\omega}\right)\vec{H} + \left(\frac{\vec{k}\cdot\vec{P}}{2\varepsilon_0^2\mu_0}\right)\vec{P} + \frac{\vec{P}\times\vec{M}}{2\varepsilon_0}$$ ∴ not always ∥ to $\vec{k}$

1.4 Suppose that we are using an optical beam of diameter D to monitor the particle content of a column of gas. For many applications we would prefer to sample as small a volume as possible, and consequently we would first choose a very small beam. But if the path length is long, a very small beam would diverge quickly and thus sample a larger cross-sectional area of the gas column. Use the uncertainty relations to derive an expression for the beam diameter to minimize the volume of gas sampled. Assume a helium/neon probing laser ($\lambda = 632.8$ nm) and a simple cone describing the convergence and divergence of the beam envelope so as to evaluate for a gas column 10 m long.

$$\frac{\Delta k_y}{k_z} = \frac{\lambda}{\pi W_0}$$

$$\Delta k_y = \frac{\lambda k_z}{\pi W_0} \qquad \Delta k_y \geq \frac{1}{2\Delta y}$$

$$\frac{\lambda k_z}{\pi W_0} \geq \frac{1}{2\Delta y}$$

$$W_0 \leq \frac{2\lambda k_z \Delta y}{\pi}$$

$$\frac{\Delta W_0}{\Delta k_z} \leq \frac{2\lambda \Delta y}{\pi} \qquad \Delta k_z \geq \frac{1}{2\Delta z} \qquad \Delta z \geq \frac{1}{2\Delta k_z}$$

$$\frac{\Delta k_y}{k_z} = \frac{\lambda}{\pi W_0} \qquad V = \frac{1}{3}\pi(\Delta y)^2 \Delta z$$

$$W_0 = \left(\frac{\lambda}{\pi \Delta k_y}\right) k_z$$

$$\frac{dW_0}{dk_z} = \frac{\lambda}{\pi \Delta k_y} = \frac{\Delta W_0}{\Delta k_z}$$

$$\frac{\Delta k_z}{\Delta k_y} = \frac{\Delta W_0 \pi}{\lambda}$$

$$\frac{\Delta k_y \Delta y \Delta k_z}{\Delta k_y} \geq \frac{\Delta W_0 \pi}{2\lambda}$$

$$\frac{\Delta y \lambda}{\Delta W_0 \pi} \geq \frac{1}{2\Delta k_z}$$

$$V = \frac{1}{3}\pi(\Delta y)^2 z$$

$$\frac{3V}{\pi} = (\Delta y)^2 z \qquad \Delta y^2 \geq \left(\frac{1}{2\Delta k_y}\right)^2 \qquad \Delta z \geq \left(\frac{1}{2\Delta k_z}\right)$$

$$\frac{3V}{\pi} \geq \frac{1}{4\Delta k_y^2 z}$$

$$\Delta k_y^2 \geq \frac{\pi}{12Vz}$$

$$\frac{\lambda^2 k_z^2}{\pi^2 W_0^2} \geq \frac{\pi}{12Vz}$$

$$\boxed{W_0^2 \leq \frac{12Vz\lambda^2 k_z^2}{\pi^3}}$$

1.5 There are various ways to specify the frequency and photon energy of a laser.
Convert the specification of photons from common coherent sources to other units.

If ν is given: $\lambda_0 = c/\nu$ $\quad \bar{\nu}(cm^{-1}) = 1/\lambda_0(cm)$ $\quad \nu = \bar{\nu} c$

from ~~to~~ nm $\quad$ ~~from~~ nm

$1\,nm = 10\,Å$ $\qquad 1\,Å = 0.1\,nm$

~~1nm = 1.23985·10^-3 eV~~ $\qquad 1\,eV = 806.549\,nm$

~~1nm = λ in nm~~ $\bar{\nu} = 1/(\lambda \cdot 10^{-9})$ $\qquad$ ~~1cm^-1 = 0.1nm~~ $\bar{\nu}(\text{in cm})$ $\lambda = (1/\bar{\nu}) \cdot 10^7$

$1\,nm = \frac{c}{\nu} = \frac{3 \cdot 10^8\,ms^{-1}}{1 \cdot 10^{-9}\,m} = 3 \cdot 10^{17}\,s^{-1}$ $\qquad 1\,Hz = 3.33 \cdot 10^{-18}\,nm$

(nm) $\lambda_0 = c/\nu = \frac{3 \cdot 10^8\,ms^{-1}\,(1 \cdot 10^9\,nm/m)}{\nu} = \frac{3 \cdot 10^{17}\,nm\,s^{-1}}{\nu}$

(Hz) $\nu = c/\lambda_0 = \frac{3 \cdot 10^{17}\,nm\,s^{-1}}{\lambda_0(nm)}$

$$\boxed{h\nu(eV) = \frac{h\nu}{q_e} = \left(\frac{6.626 \cdot 10^{-34}\,J\,s\,eV}{1.602 \cdot 10^{-19}\,J}\right)\nu(Hz)}$$

Source	eV	λ(Å)	λ(nm)	ν(Hz)	$\bar{\nu}(cm^{-1})$
GaAs	1.47	~~11856~~ 8441	1185.6 ~~844.08~~	~~2.53·10^14~~ $3.554 \cdot 10^{14}$	~~8434.5~~ 11847
Ar^+	2.41	5145	514.5	$5.83 \cdot 10^{14}$	19436.3
He:Ne	1.96	6328	632.8	$4.74 \cdot 10^{14}$	15803
CO_2	0.117	106045 ~~9443~~	10604.5 ~~944.3~~	$2.82 \cdot 10^{13}$	943
ISM Band	~~2.24·10^-7~~ $5.608 \cdot 10^{-8}$	22.12 m	$2.212 \cdot 10^{10}$	13.56 MHz	$4.52 \cdot 10^{-4}$
KrF	4.96	2490	249	$1.2 \cdot 10^{15}$	$4.02 \cdot 10^{4}$

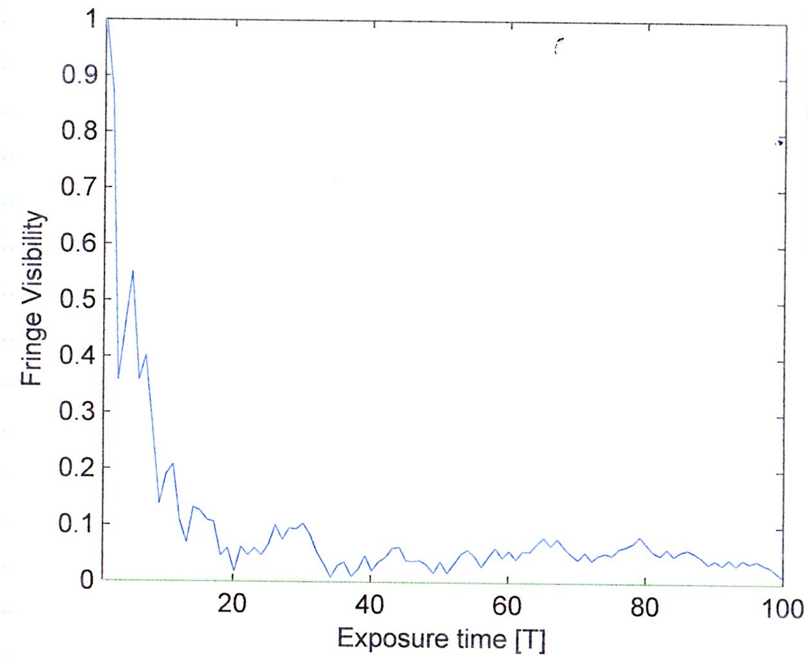
1.6 Repeat the coin-flipping routine of Section 1.11, and find the fringe visibility as a function of exposure time.

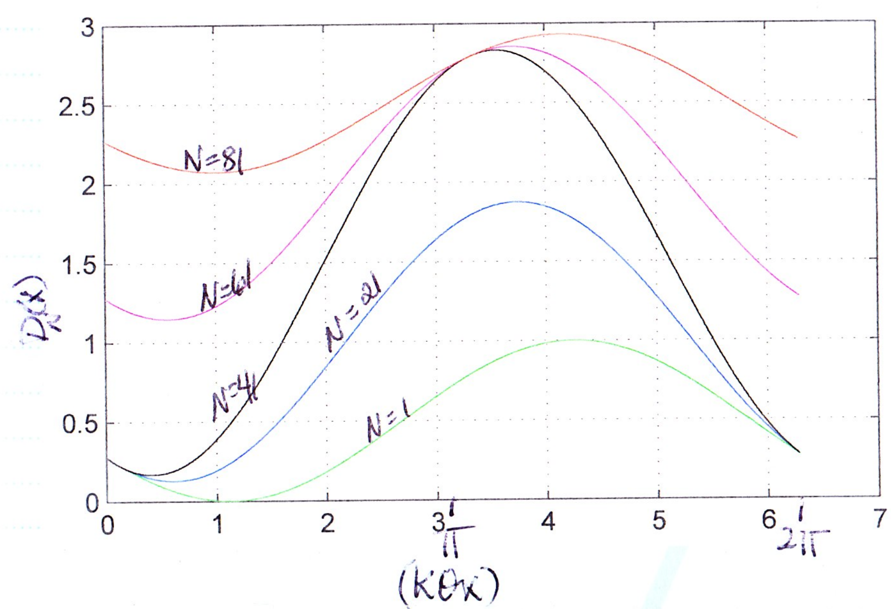
$$D_N(x) = \frac{1}{NT}\Big[I_{max}T\cos^2\Big(\frac{k\theta x+\phi_0}{2}\Big) + I_{max}T\cos^2\Big(\frac{k\theta x+\phi_0+\phi_1}{2}\Big) + I_{max}T\cos^2\Big(\frac{k\theta x+\phi_0+\phi_1+\phi_2}{2}\Big) + \cdots\Big]$$

Penny	Nickel	Dime	$\Delta\phi$
H	H	H	0
H	H	T	+45°
H	T	H	+90°
T	H	H	135°
T	T	T	180°
T	T	H	-45°
T	H	T	-90°
H	T	T	-135°

```
function Lasers_Ch1_Pr6
x = 0:0.01:2.0*pi;
N=100;V=zeros(N);dN=20;
coins = fix(rand(N,1)*8+1);
jump=[0,45,90,135,180,-45,-90,-135];
colors=['r','g','b','k','m'];
phase=jump(coins)*180/pi;
D = 0;arg = 0;j=0;figure(1);
ylabel('Density of Exposure');
xlabel('Exposure time [T]');
for i = 1:dN:N
   arg = arg + phase(i);j=j+1;
   D = D + (cos((x+arg)/2)).^2;
   Visibility = (max(D)-min(D))/(max(D)+min(D));
   V(i) = Visibility;
   plot(x,D,colors(mod(j,5)+1)),grid on;hold on;
end
figure(2);
plot(V);axis([1 N 0 1]);
ylabel('Fringe Visibility');
xlabel('Exposure time [T]');
```

Can compute the fringe visibility by determining the density of the exposure of the film by assuming that the shutter is open for $t \geq T$ sec and $t \leq NT$ sec.





1.7 Show that the Fourier transform of the field given by

$$E(y) = E_0 e^{-(y/w_0)^2}$$

is $E(k_y) = \pi^{1/2} w_0 E_0 e^{-(k_y w_0/2)^2}$

definition of Fourier Transform $\left.\right\} \mathcal{F}(f(x)) = F(\alpha) = \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x} dx$

$$E(k_y) = \int_{-\infty}^{+\infty} E(y) e^{-ik_y y} dy = \int_{-\infty}^{+\infty} E_0 e^{-(y/w_0)^2} e^{-ik_y y} dy$$

$$E(k_y) = E_0 \int_{-\infty}^{+\infty} e^{-y^2/w_0^2 - ik_y y} dy$$

complete the square: $A = iw_0/2$

$$\left(\frac{y}{w_0}\right)^2 + ik_y y + A^2 k_y^2 - A^2 k_y^2 = \left(\frac{y}{w_0} + \frac{iw_0 k_y}{2}\right)^2 + \frac{w_0^2 k_y^2}{2}$$

$$E(k_y) = E_0 \int_{-\infty}^{+\infty} e^{-\left(\frac{y}{w_0} + \frac{iw_0 k_y}{2}\right)^2 + \frac{w_0 k_y^2}{2}} dy$$

$$E(k_y) = E_0 e^{-(w_0^2 k_y^2/2^2)} \int_{-\infty}^{\infty} e^{-(y/w_0 + iw_0 k_y/2)^2} dy$$

let $u = \frac{y}{w_0} + \frac{iw_0 k_y}{2}$

$du = \frac{dy}{w_0}$

$$E(k_y) = E_0 e^{-(w_0 k_y/2)^2} \int_{-\infty}^{+\infty} w_0 e^{-u^2} du$$

$$E(k_y) = E_0 w_0 e^{-(w_0 k_y/2)^2} \int_{-\infty}^{+\infty} e^{-u^2} du$$

$$E(k_y) = \sqrt{\pi}\, w_0 E_0 e^{-(k_y w_0/2)^2}$$

1.8 The $TEM_{0,0}$ Gaussian beam has the smallest value of the product $\Delta x \Delta k_x = 1/2$ allowed by the uncertainty relationship. The quantities Δx and Δk_x are interpreted to be as

$$\Delta x^2 = \frac{\int x^2 |E(x)|^2 dx}{\int |E(x)|^2 dx} \qquad \Delta k_x^2 = \frac{\int k_x^2 |E(k_x)|^2 dk_x}{\int |E(k_x)|^2 dx}$$

with $E(x)$ and $E(k_x)$ being related by the Fourier transform.

a) What are the values for Δx and Δk_x for
$E(x) = E_0 e^{-(x/w_0)^2}$ (ie $TEM_{0,0}$)?
$E(k_x) = \sqrt{\pi}\, w_0 E_0 e^{-(k_x w_0/2)^2}$

$$\Delta x^2 = \frac{\int E_0^2 x^2 e^{-2(x/w_0)^2} dx}{\int E_0^2 e^{-2(x/w_0)^2} dx} = \frac{\cancel{E_0^2} \int x^2 e^{-(\sqrt{2}x/w_0)^2} dx}{\cancel{E_0^2} \int e^{-(\sqrt{2}x/w)^2} dx}$$

$$u = \frac{\sqrt{2}\,x}{w_0} \qquad du = \frac{\sqrt{2}\,dx}{w_0} \Rightarrow dx = \frac{w_0\,du}{\sqrt{2}} \qquad x = \frac{w_0 u}{\sqrt{2}}$$

$$\Delta x^2 = \frac{w_0/\sqrt{2}}{w_0/\sqrt{2}} \frac{\int (w_0^2/2) u^2 e^{-u^2} du}{\int e^{-u^2} du} = \frac{w_0^2}{2} \frac{\int u^2 e^{-u^2} du}{\int e^{-u^2} du}$$

$$\Delta x^2 = \frac{w_0^2}{2} \frac{\Gamma(\frac{3}{2})}{\sqrt{\pi}} = \frac{w_0^2 \cancel{\frac{1}{2}} \Gamma(\frac{1}{2})}{2\sqrt{\pi}} = \frac{w_0^2}{4} \frac{\sqrt{\pi}}{\sqrt{\pi}} = \frac{w_0^2}{4}$$

$$\Rightarrow \boxed{\Delta x^2 = w_0^2/4}$$

$$\Delta k_x^2 = \frac{\int \pi w_0^2 E_0^2 k_x^2 e^{-2(k_x w_0/2)^2} dk_x}{\int \pi w_0^2 E_0^2 e^{-2(k_x w_0/2)^2} dk_x} = \frac{\int k_x^2 e^{-2(k_x w_0/2)^2} dk_x}{\int e^{-2(k_x w_0/2)^2} dk_x}$$

$$u = \frac{\sqrt{2}\,k_x w_0}{2} \qquad du = \frac{\sqrt{2}\,w_0\,dk_x}{2} \qquad dk_x = \frac{2\,du}{\sqrt{2}\,w_0} \qquad k_x = \frac{2u}{\sqrt{2}\,w_0}$$

$$\Delta k_x^2 = \frac{\frac{2}{w_0^2} \int u^2 e^{-u^2} du \cancel{\left(\frac{2}{\sqrt{2} w_0}\right)}}{\int e^{-u^2} du \cancel{\left(\frac{2}{\sqrt{2} w_0}\right)}} = \frac{2}{w_0^2} \frac{\int u^2 e^{-u^2} du}{\int e^{-u^2} du}$$

$$\Delta k_x^2 = \frac{2}{w_0^2} \frac{\Gamma(3/2)}{\sqrt{\pi}} = \frac{2}{w_0^2} \frac{\frac{1}{2}\Gamma(1/2)}{\sqrt{\pi}} = \frac{1}{w_0^2} \frac{\sqrt{\pi}}{\sqrt{\pi}} = \frac{1}{w_0^2}$$

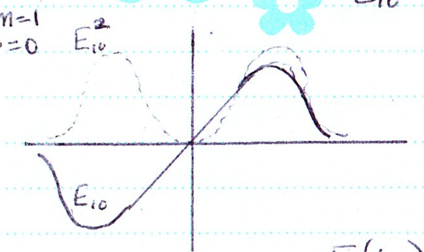
$$\boxed{\Delta k_x^2 = \frac{1}{w_0^2}}$$

$$\Delta x^2 \Delta k_x^2 = \frac{\cancel{w_0^2}}{4} \frac{1}{\cancel{w_0^2}} = \frac{1}{4}$$

$$\Rightarrow \Delta x \Delta k_x = 1/2$$

b) What is the uncertainty product for a field given by

$$E_{10} = \frac{\sqrt{2}\,x}{W} e^{-(x^2+y^2)/w_0^2}$$

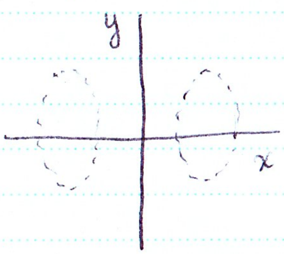


$$E_{10} = \frac{\sqrt{2}\, r\cos\phi}{W} e^{-(r/w_0)^2}$$

$$E(k_r) = \int_{-\infty}^{+\infty} E(r)\, e^{-ik_r r}\, dr$$

$x = r\cos\phi$

$x^2 + y^2 = r^2$



$$(\Delta y)^2 = \frac{\int \left(\frac{E_{10}\sqrt{2}\,x}{W^2}\right)^2 e^{-2(x^2+y^2)/w_0^2}\, dx\, dy}{\int \frac{E_{10}^2(\sqrt{2}x)^2}{W^2} e^{-2(x^2+y^2)/w_0^2}\, dy}$$

$$(\Delta y)^2 = \frac{E_{10}^2\, 2x^2 e^{-2x^2/w_0^2}\, W^{-2}}{E_{10}^2\, 2x^2 e^{-2x^2/w_0^2}\, W^{-2}} \frac{\int y^2 e^{-2y^2/w_0^2}\, dy}{\int e^{-2y^2/w_0^2}\, dy}$$

$$(\Delta y)^2 = \frac{\int y^2 e^{-2y^2/w_0^2}\, dy}{\int e^{-2y^2/w_0^2}\, dy}$$

$u = \frac{\sqrt{2}}{w_0} y \quad du = \frac{\sqrt{2}}{w_0} dy \quad dy = \frac{w_0}{\sqrt{2}} du \quad y = \frac{w_0 u}{\sqrt{2}}$

$$(\Delta y)^2 = \frac{\int \left(\frac{w_0^2 u^2}{2}\right) e^{-u^2} \frac{w_0}{\sqrt{2}}\, du}{\int e^{-u^2} \frac{w_0}{\sqrt{2}}\, du} = \frac{w_0^2}{2} \frac{\int u^2 e^{-u^2}\, du}{\int e^{-u^2}\, du}$$

$$(\Delta y)^2 = \frac{w_0^2}{2} \frac{\Gamma(3/2)}{\sqrt{\pi}} = \frac{w_0^2\, \frac{1}{2}\Gamma(1/2)}{2\sqrt{\pi}} = \frac{w_0^2 \sqrt{\pi}}{4\sqrt{\pi}} = \frac{w_0^2}{4}$$

$$(\Delta y)^2 = \frac{w_0^2}{4} \Rightarrow \boxed{\Delta y = \frac{w_0}{2}}$$

$$(\Delta x)^2 = \frac{\frac{2}{W^2} e^{-2y^2/w_0^2} \int x^2 x^2 e^{-2x^2/w_0^2}\, dx}{\frac{2}{W^2} e^{-2y^2/w_0^2} \int x^2 e^{-2x^2/w_0^2}\, dx} = \frac{\int x^4 e^{-2x^2/w_0^2}\, dx}{\int x^2 e^{-2x^2/w_0^2}\, dx}$$

$u = \frac{\sqrt{2}}{w_0} x \quad du = \frac{\sqrt{2}}{w_0} dx \quad dx = \frac{w_0}{\sqrt{2}} du \quad \cancel{dx} \Rightarrow x = \frac{w_0 u}{\sqrt{2}}$

$$(\Delta x)^2 = \frac{\int \frac{w_0^4}{4} u^4 e^{-u^2} \frac{w_0}{\sqrt{2}}\, du}{\int \frac{w_0^2}{2} u^2 e^{-u^2} \frac{w_0}{\sqrt{2}}\, du} = \frac{w_0^2}{2} \frac{\int u^4 e^{-u^2}\, du}{\int u^2 e^{-u^2}\, du}$$

$$(\Delta x)^2 = \frac{w_0^2}{2} \frac{\Gamma(5/2)}{\Gamma(3/2)} = \frac{w_0^2\, \frac{3}{2}\Gamma(3/2)}{2\, \frac{1}{2}\Gamma(1/2)} = \frac{w_0^2 (3/2)(1/2)\Gamma(1/2)}{2(1/2)\Gamma(1/2)}$$

$$(\Delta x)^2 = \frac{3 w_0^2}{4} \Rightarrow \boxed{\Delta x = \frac{\sqrt{3}\, w_0}{2}}$$

$$E(k_y) = \int_{-\infty}^{+\infty} \frac{\sqrt{2}\,x}{W} e^{-x^2/w_0^2} e^{-y^2/w_0^2} e^{-ik_y y}\, dy$$

$$E(k_y) = \frac{\sqrt{2}\,x\, e^{-x^2/w_0^2}}{W} \int_{-\infty}^{+\infty} e^{-(y^2/w_0^2 - ik_y y)}\, dy$$

$$E(k_y) = \frac{\sqrt{2}\, x\, e^{-x^2/w_0^2}}{w} \int_{-\infty}^{+\infty} e^{-\left((y/w_0)^2 + ik_y y\right)} dy$$

complete the square: $A = i w_0/2$

$$\left(\frac{y}{w_0}\right)^2 + ik_y y + A^2 k_y^2 - A^2 k_y^2 = \left(\frac{y}{w_0}\right)^2 + \frac{2i k_y y}{2} + \left(\frac{w_0}{2}\right)^2 k_y^2 + \left(\frac{w_0 k_y}{2}\right)^2$$

$$= \left(\frac{y}{w_0} + \frac{i w_0 k_y}{2}\right)^2 + \left(\frac{w_0 k_y}{2}\right)^2$$

$$E(k_y) = \frac{\sqrt{2}\, x e^{-x^2/w_0^2}}{w} \int_{-\infty}^{+\infty} e^{-(y/w_0 + i w_0 k_y/2)^2} e^{-(w_0 k_y/2)^2} dy$$

$$E(k_y) = \frac{\sqrt{2}\, x e^{-(x^2/w_0^2 + w_0^2 k_y^2/4)}}{w} \int_{-\infty}^{+\infty} e^{-(y/w_0 + i w_0 k_y/2)^2} dy$$

let $u = \left(\frac{y}{w_0} + \frac{i w_0 k_y}{2}\right)$ $\quad du = \frac{1}{w_0} dy \quad dy = w_0\, du$

$$E(k_y) = \frac{\sqrt{2}\, x e^{-((x/w_0)^2 + (w_0 k_y/2)^2)}}{w} \int_{-\infty}^{+\infty} e^{-u^2} w_0\, du$$

$$E(k_y) = \sqrt{2}\, x \left(\frac{w_0}{w}\right) e^{-((x/w_0)^2 + (w_0 k_y/2)^2)} \sqrt{\pi}$$

$$E(k_y) = \sqrt{2\pi}\left(\frac{w_0}{w}\right) x e^{-((x/w_0)^2 + (w_0 k_y/2)^2)} = E_{0yk_y} e^{-(w_0 k_y/2)^2}$$

where $E_{0yk_y} = \sqrt{2\pi}\left(\frac{w_0}{w}\right) x e^{-(x/w_0)^2}$

$$(\Delta k_y)^2 = \frac{\int |E(k_y)|^2 k_y^2\, dk_y}{\int |E(k_y)|^2 dk_y} = \frac{E_{0yk_y}^2 \int k_y^2 e^{-2(w_0 k_y/2)^2} dk_y}{E_{0yk_y}^2 \int e^{-2(w_0 k_y/2)^2} dk_y}$$

$$(\Delta k_y)^2 = \frac{\int k_y^2 e^{-(\sqrt{2} w_0 k_y/2)^2} dk_y}{\int e^{-(\sqrt{2} w_0 k_y/2)^2} dk_y}$$

$k_y = 2u/\sqrt{2} w_0$

$u = \sqrt{2} w_0 k_y/2 \quad du = \sqrt{2} w_0\, dk_y/2 \quad dk_y = 2du/\sqrt{2} w_0$

$$(\Delta k_y)^2 = \frac{\int (2u^2/w_0^2) e^{-u^2} du\, (2/\sqrt{2} w_0)}{\int e^{-u^2} du\, (2/\sqrt{2} w_0)} = \frac{2 \int u^2 e^{-u^2} du}{w_0^2 \int e^{-u^2} du}$$

$$(\Delta k_y)^2 = \frac{2}{w_0^2} \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{\sqrt{\pi}} = \frac{2}{w_0^2} \frac{(\frac{1}{2})\sqrt{\pi}}{\sqrt{\pi}} = \frac{1}{w_0^2}$$

$$(\Delta k_y)^2 = \frac{1}{w_0^2} \Rightarrow \boxed{\Delta k_y = 1/w_0}$$

$$\Delta k_y \Delta y = \left(\frac{1}{w_0}\right)\left(\frac{w_0}{2}\right) = \frac{1}{2} \Rightarrow \boxed{\Delta k_y \Delta y = 1/2}$$

$$E(k_x) = \int_{-\infty}^{+\infty} E(x) e^{-ik_x x} dx = \frac{\sqrt{2}}{w} e^{-(y/w_0)^2} \int_{-\infty}^{+\infty} e^{-(x^2/w_0^2 + ik_x x)} x\, dx$$

complete square with $A = i w_0/2$

$$E(k_x) = \frac{\sqrt{2}}{w} e^{-[(y/w_0)^2 + (w_0 k_x/2)^2]} \int_{-\infty}^{+\infty} x e^{-(x/w_0 + i w_0 k_x/2)^2} dx$$

$u = \frac{x}{w_0} + \frac{i k_x w_0}{2} \quad du = \frac{dx}{w_0} \quad dx = w_0\, du \quad x = \left(w_0 u - \frac{i k_x w_0^2}{2}\right)$

$$E(k_x) = \frac{\sqrt{2}}{w} e^{-[(y/w_0)^2 + (w_0 k_x/2)^2]} \int_{-\infty}^{+\infty} \left(w_0 u - \frac{i k_x w_0^2}{2}\right) e^{-u^2} w_0\, du$$

$$E(k_x) = \sqrt{2}\left(\frac{w_0}{w}\right) e^{-[(y/w_0)^2 + (w_0 k_x/2)^2]} \left\{ \underbrace{\int_{-\infty}^{+\infty} w_0 u e^{-u^2} du}_{0} + \int_{-\infty}^{+\infty} \frac{i k_x w_0^2}{2} e^{-u^2} du \right\}$$

$$E(k_x) = \sqrt{2}\left(\frac{w_0}{w}\right) e^{-[(y/w_0)^2 + (w_0 k_x/2)^2]} \left\{ \frac{-w_0^2 i k_x \sqrt{\pi}}{2} \right\}$$

$$E(k_x) = \left(-\frac{\sqrt{2\pi}}{2} \frac{w_0^3}{w} i e^{-(y/w_0)^2}\right) k_x e^{-(w_0 k_x/2)^2} = E_{0} k_x e^{-(w_0 k_x/2)^2}$$

$$E_{0} = -\frac{\sqrt{2\pi}}{2} i \frac{w_0^3}{w} e^{-(y/w_0)^2}$$

$$(\Delta k_x)^2 = \frac{\int |E(k_x)|^2 k_x^2\, dk_x}{\int |E(k_x)|^2 dk_x} = \frac{E_{0}^2}{E_{0}^2} \frac{\int k_x^2 k_x^2 e^{-2(w_0 k_x/2)^2} dk_x}{\int k_x^2 e^{-2(w_0 k_x/2)^2} dk_x}$$

$$(\Delta k_x)^2 = \frac{\int_{-\infty}^{+\infty} k_x^4 e^{-(\sqrt{2} w_0 k_x/2)^2} dk_x}{\int_{-\infty}^{\infty} k_x^2 e^{-(\sqrt{2} w_0 k_x/2)^2} dk_x}$$

Let $u = \frac{\sqrt{2} w_0 k_x}{2}$ $\quad du = \frac{\sqrt{2} w_0 dk_x}{2}$ $\quad dk_x = \frac{2du}{\sqrt{2} w_0}$ & $k_x = \frac{2u}{\sqrt{2} w_0}$

$$(\Delta k_x)^2 = \frac{\int_{-\infty}^{+\infty} \frac{2^4 u^4}{2^2 w_0^4} e^{-u^2} \frac{2du}{\sqrt{2} w_0}}{\int_{-\infty}^{+\infty} \frac{2^2 u^2}{2 w_0} e^{-u^2} \frac{2du}{\sqrt{2} w_0}} = \frac{\int_{-\infty}^{+\infty} \frac{4}{w_0^2} u^4 e^{-u^2} du}{\int_{-\infty}^{+\infty} \frac{2}{w_0} u^2 e^{-u^2} du}$$

$$(\Delta k_x)^2 = \frac{2}{w_0^2} \frac{\int u^4 e^{-u^2} du}{\int u^2 e^{-u^2} du} = \frac{2}{w_0^2} \frac{\Gamma(5/2)}{\Gamma(3/2)} = \frac{2 (3/2) \Gamma(3/2)}{w_0^2 (1/2) \Gamma(1/2)}$$

$$(\Delta k_x)^2 = \frac{2 (3/2)(1/2) \Gamma(1/2)}{w_0^2 (1/2) \Gamma(1/2)} = \frac{3}{w_0^2}$$

$$\boxed{(\Delta k_x) = \frac{\sqrt{3}}{w_0}}$$

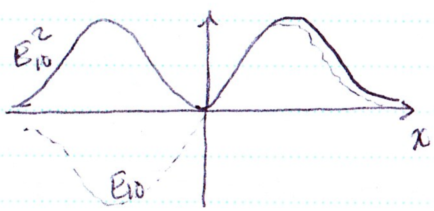
$$\Delta k_x \Delta x = \frac{\sqrt{3}}{w_0} \frac{\sqrt{3}\, w_0}{2} = \frac{3}{2}$$

1.8 (c) Sketch the intensity $E_{10} E_{10}^* / 2\eta_0$ as a function of x.

$$\eta_0 = \left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2} \sim 377\Omega$$

$$E_{10} = \frac{\sqrt{2}\, x}{w} e^{-(x^2+y^2)/w_0^2}$$

$$I = \frac{E_{10} E_{10}^*}{2\eta_0} = \frac{2 x^2 e^{-(x^2+y^2)/w_0^2}}{2\eta_0 w^2} = \left(\frac{1}{\eta_0 w^2}\right) x^2 e^{-(x^2+y^2)/w_0^2}$$



1.9 Show that the factor 2 belongs in (1.10.4).

$$V_{out} = E_T^2 = 4E_0^2 \sin^2\left(kz + \Delta\phi(t)/2\right)$$

$$E_T = E^+ + E^- = E_0 e^{-j(kz+\Delta\phi(t))} - E_0 e^{+jkz}$$

$$\cancel{E_T = E_0(\cos(kz+\Delta\phi(t))) - E_0 \cos(kz)}$$

$$\cancel{E_T = E_0 \cos(kz+\Delta\phi(t)) - E_0 \cos(kz)}$$

$$E_T^* E_T = \left(E_0 e^{j(kz+\Delta\phi)} - E_0 e^{-jkz}\right)\left(E_0 e^{-j(kz+\Delta\phi)} - E_0 e^{jkz}\right)$$

$$= \cancel{E_0^2 - E_0^2 e^{+2jkz+j\Delta\phi} - E_0^2 e^{-2jkz-j\Delta\phi} + E_0^2}$$

$$\cancel{= 2E_0^2 - E_0^2 e} = \cancel{E_0^2 - E_0^2 e^{2(jkz+j\Delta\phi/2)} + E_0^2 - E_0^2 e^{-2(jkz+j\Delta\phi/2)}}$$

$$= \cancel{E_0^2(1-\cos^2(kz+\tfrac{\Delta\phi}{2})) + E_0^2(1-\cos^2(kz+\tfrac{\Delta\phi}{2}))}$$

$$\cancel{= 2E_0^2 \sin^2(kz+\Delta\phi)}$$

$$= \cancel{E_0^2\{\sin^2(kz+\Delta\phi) + \sin^2(kz+\Delta\phi)\}}$$

$$= \cancel{E_0^2\{\sin^2(kz+\Delta\phi) + \sin^2(kz-\Delta\phi) + 2\sin(kz+\Delta\phi)\sin(kz-\Delta\phi) - 2\sin(kz+\Delta\phi)\sin(kz-\Delta\phi)\}}$$

$$= \cancel{E_0^2\{(\sin(kz+\Delta\phi)+\sin(kz-\Delta\phi))^2 - 2\sin(kz+\Delta\phi)\sin(kz-\Delta\phi)\}}$$

$$E_T^* E = E_0^2 - E_0^2 e^{+2j(kz+\Delta\phi/2)} - E_0^2 e^{-2j(kz+\Delta\phi/2)} + E_0^2$$

$$= 2E_0^2 - E_0^2\left(e^{+2j(kz+\Delta\phi/2)} + e^{-2j(kz+\Delta\phi/2)}\right)$$

$$= 2E_0^2 - 2E_0^2 \cos[2(kz+\Delta\phi/2)] = 2E_0^2\{1-\cos 2(kz+\Delta\phi/2)\}$$

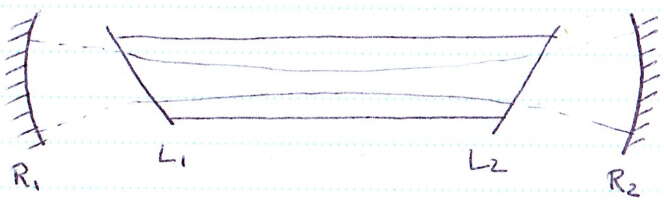
$$= 2E_0^2(1 - 1 + 2\sin^2(kz+\Delta\phi/2))$$

use $\cos 2A = 1 - 2\sin^2 A$

$$= 4E_0^2 \sin^2(kz + \Delta\phi/2)$$

$2\cos\theta = e^{i\theta} + e^{-i\theta}$

.10 Quartz windows oriented at Brewster's angle are commonly used for He:Ne lasers in the manner indicated in Figure 0.1. What is the angle from the axis of the cavity? (η[quartz] $\approx$ 1.43).

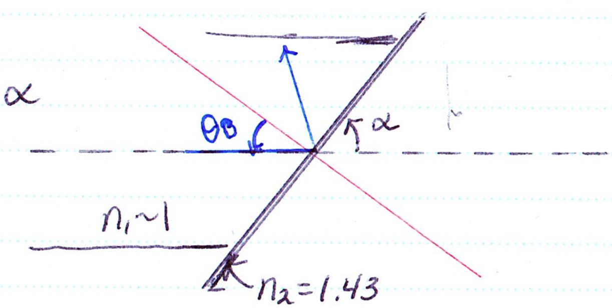


$\tan\theta_B = \frac{n_2}{n_1}$ Brewster angle

$\theta_B = \tan^{-1} 1.5 = .98 \text{rad} = 56.3°$

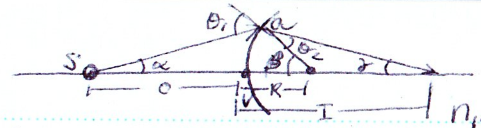
$\theta_B = \tan^{-1}\left(\frac{1}{1.5}\right) = .588 \text{rad} = 33.69°$

$\theta_B = 90° - \alpha$



$\theta_B = \tan^{-1}\frac{n_2}{n_1} = \tan^{-1}\frac{1.43}{1} = 55.034°$

$\alpha = 90 - \theta_B \approx 35°$



$\theta_1 = \alpha + \beta \quad \beta = \theta_2 + \gamma \quad \alpha \approx \frac{aV}{O}$

$n_1 \sin\theta_1 = n_2 \sin\theta_2 \quad \beta \approx aV/R$

$n_1\theta_1 \sim n_2\theta_2 \quad \gamma \approx aV/I$

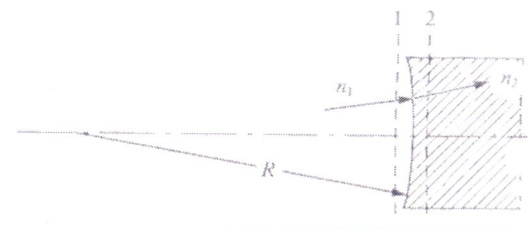
$n_1\alpha + n_2\gamma = (n_2 - n_1)\beta$

2.1 Derive the ray matrix for a ray entering a spherical dielectric interface.

$\frac{n_1}{O} + \frac{n_2}{I} = \frac{n_2 - n_1}{R}$

$\frac{aV}{O} + \frac{n_2 aV}{n_1 I} = \frac{\left(\frac{n_2}{n_1} - 1\right)aV}{R}$

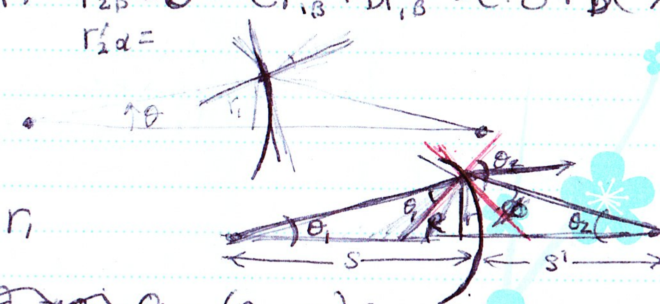
$\alpha + \frac{n_2}{n_1}\gamma = \frac{\beta\left(\frac{n_2}{n_1} - 1\right)}{R}$



$r'_{1\beta} = -r_2/R$

through #1: $r'_{2\beta} = 0 = C r_{1\beta} + D r'_{1\beta} = C \cdot 0 + D(-r_2/R)$

$r'_{2\alpha} =$



$r_2 = r_1$

~~$n_2 \sin\theta_2 = n_1 \sin\theta_1 - (n_2 - n_1)\frac{r_V}{R}$~~

$R\sin\theta = (R + s)\sin\theta_1 \qquad R\sin\phi = (s' - r)\sin\theta_2$

$n_1 \sin\theta = n_2 \sin\phi$

$\Rightarrow \frac{\sin\theta_1}{\sin\theta_2} = \left(\frac{s'-r}{s+r}\right)\frac{n_2}{n_1} \xrightarrow[\text{approx}]{\text{paraxial}} \frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}$

$\sin\theta \sim \theta$

$r_2 = r_1 \qquad \theta_1 = r'_1 = \frac{r_1}{s} \quad \theta_2 = \frac{-r_2}{s'} = r'_2$

$r'_2 = r'_1\left(\frac{n_1}{n_2}\right) - \left(1 - \frac{n_1}{n_2}\right)\frac{r_2}{R} \Rightarrow \begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n_1}{n_2} - 1\right) & \frac{n_1}{n_2} \end{bmatrix}$